MATH 215
Mathematical Analysis

Lecture Notes

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Contents

1 The Real And Complex Number Systems

1.1 The Real Number System

$\mathbb{R}$: The set of real numbers

$\mathbb{Q}$: The set of rational numbers (Numbers in the form $\frac{m}{n}$ where m, n are integers, $n \neq 0$)

$\mathbb{Z}$: The set of integers, $\{0, 1, -1, 2, -2, 3, -3, \ldots\}$

$\mathbb{N}$: The set of natural numbers, $\{1, 2, 3, 4, \ldots\}$

Proofs:

1) Direct Proofs
2) Indirect Proofs
   a) Proof by Contraposition
   b) Proof by Contradiction

1.1 Example (Direct Proof)**.** If f is differentiable at $x = c$ then f is continuous at $x = c$.

Proof. Hypothesis: f is differentiable at $x = c$, i.e.

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) \quad \text{exists and it is a number.}$$

Claim: f is continuous at c, i.e. $\lim_{x \to c} f(x) = f(c)$.

$$\begin{aligned} \lim_{x \to c} f(x) &= \lim_{x \to c} \left(\frac{f(x) - f(c)}{x - c}(x - c) + f(c) \right) \\ &= \underbrace{\lim_{x \to c} \frac{f(x) - f(c)}{x - c}}_{f'(c)} \underbrace{\lim_{x \to c}(x - c)}_{0} + \underbrace{\lim_{x \to c} f(c)}_{f(c)} \\ &= f(c) \end{aligned}$$

□

1.2 Example (Indirect Proof: Proof by Contraposition)**.** Let n be an integer. If $\underbrace{n^2 \text{ is even}}_{p}$ then $\underbrace{n \text{ is even}}_{q}$.

If p then q: $p \Rightarrow q$. This is equivalent to: If not q then not p, i.e. $\sim q \Rightarrow \sim p$. If n is odd then n^2 is odd.

Proof. Assume n is odd then $n = 2k + 1$ for some integer k. Then

$$n^2 = 4k^2 + 4k + 1 = 2(\underbrace{2k^2 + 2k}_{\text{an integer}}) + 1$$

So n^2 is odd. □

1.3 Example (Indirect Proof: Proof by Contradiction)**.** Show that $\sqrt{2}$ is not a rational number.

Proof. If $\underbrace{c^2 = 2}_{p}$ then $\underbrace{c \text{ is not rational}}_{q}$. We assume p and $\sim q$, proceed and get a contradiction. So assume $c^2 = 2$ and c is rational. Then $c = \frac{m}{n}$ where $m, n \in \mathbb{Z}$, $n \neq 0$ and m, n have no common factors. We have $c^2 = \frac{m^2}{n^2}$ then $m^2 = 2n^2$. So m^2 is even and m is even. Then $m = 2k$ for some integer k. We get $4k^2 = 2n^2$ or $2k^2 = n^2$. Then n^2 is even and n is even. Then $n = 2\ell$ for some integer ℓ. So m and n have a common factor. A contradiction. □

Some Symbols And Notation

- $p \Rightarrow q$: if p then q (p implies q)
- $p \Leftrightarrow q$: if p then q and if q then p (p iff q)
- $\ni$: such that
- $\therefore$: therefore, so
- $\forall$: for all, for every
- $\exists$: for some, there exists

1.4 Example. Consider the following two statements:

(1) $\forall x \in \mathbb{R} \quad \exists y \in \mathbb{R}, \quad y < x$

(2) $\exists y \in \mathbb{R} \quad \forall x \in \mathbb{R}, \quad y < x \quad \longrightarrow$ not true

(1) says, given any real number x we can find a real number y (depending on x) such that y is smaller than x.

(2) says, there is a real number y that is smaller than every real number x. This is false.

1.5 Example. Let $A = \{p : p \in \mathbb{Q},\ p > 0,\ p^2 < 2\}$ and $B = \{p : p \in \mathbb{Q},\ p > 0,\ p^2 > 2\}$.

Claim: A has no largest element and B has no smallest element.

Proof. Given any $p \in \mathbb{Q}$, $p > 0$, let $q = p - \frac{p^2-2}{p+2} = \frac{2p+2}{p+2}$. Then $q \in \mathbb{Q}$ and $q > 0$. Let $p \in A$, i.e. $p^2 < 2$. Then show $q \in A$ and $p < q$.

$$p^2 - 2 < 0 \quad \text{so} \quad q = p - \frac{p^2-2}{p+2} > p$$

$$q^2 - 2 = \left(\frac{2p+2}{p+2}\right)^2 - 2 = \frac{4p^2 + 4 + 8p - 2p^2 - 8 - 8p}{p^2 + 4 + 4p} = \frac{2(p^2-2)}{(p+2)^2} < 0$$

So $q^2 < 2$. Then we have $q \in A$ and A has no largest element. $\square$

Properties of $\mathbb{R}$

1) $\mathbb{R}$ has two operations $+$ and $\cdot$ with respect to which it is a field.

(i) $\forall x, y \in \mathbb{R}, \quad x + y \in \mathbb{R}$

(ii) $\forall x, y \in \mathbb{R}, \quad x + y = y + x$ (commutativity of $+$)

(iii) $\forall x, y, z \in \mathbb{R}, \quad x + (y + z) = (x + y) + z$ (associativity of $+$)

(iv) $\exists$ an element (0 element) such that $\forall x \in \mathbb{R}, \quad x + 0 = x$

(v) $\forall x \in \mathbb{R} \quad \exists$ an element $-x \in \mathbb{R}$ such that $x + (-x) = 0$

(vi) $\forall x, y \in \mathbb{R}, \quad x \cdot y \in \mathbb{R}$

(vii) $\forall x, y \in \mathbb{R}, \quad x \cdot y = y \cdot x$

(viii) $\forall x, y, z \in \mathbb{R}, \quad x \cdot (y \cdot z) = (x \cdot y) \cdot z$

(ix) $\exists$ an element $1 \neq 0$ in $\mathbb{R}$ such that $\forall x \in \mathbb{R}, \quad x \cdot 1 = x$

(x) $\forall x \neq 0$ in $\mathbb{R}$ there is an element $\frac{1}{x}$ in $\mathbb{R}$ such that $x \cdot \frac{1}{x} = 1$

(xi) $\forall x, y, z \in \mathbb{R}, \quad x \cdot (y + z) = x \cdot y + x \cdot z$

1.6 Remark. Note that $\mathbb{Q}$ with $+$ and $\cdot$ is also a field.

2) $\mathbb{R}$ is an ordered field, i.e. there is a relation $<$ with the following properties

(i) $\forall x, y \in \mathbb{R}$ one and only one of the following is true:

$$x < y, \quad x = y, \quad y < x \quad \text{(Trichotomy Law)}$$

(ii) $\forall x, y, z \in \mathbb{R}, \quad x < y$ and $y < z \Rightarrow x < z \quad$ (Transitive Law)

(iii) $\forall x, y, z \in \mathbb{R}, \quad x < y \Rightarrow x + z < y + z$

(iv) $0 < x$ and $0 < y \Rightarrow 0 < x \cdot y$

1.7 Remark. Note that $\mathbb{Q}$ is also an ordered field.

3) $\mathbb{R}$ is complete.

1.8 Definition. Let $E \subset \mathbb{R}$. We say E is *bounded above* if there is an element $b \in \mathbb{R}$ such that $\forall x \in E$ we have $x \leq b$. b is called an *upper bound for E.*

1.9 Example. $E = \{p : p \in \mathbb{Q},\ p > 0,\ p^2 < 2\}$. Then $b = \frac{3}{2}$, $b = 2$, $b = 5$, $b = 100$, $b = \sqrt{2}$ are all upper bounds for E.

1.10 Example. $E = \mathbb{N} = \{1, 2, 3, 4, \ldots\}$ is not bounded above.

1.11 Remark. *Bounded below* and *lower bound* are defined analogously.

1.12 Definition. Let $E \subset \mathbb{R}$ be bounded above. A number $b \in \mathbb{R}$ is called a *least upper bound* (lub) or *supremum* (sup) of E if

(i) b is an upper bound for E

and

(ii) if b' any upper bound for E, then $b \leq b'$

1.13 Remark. $\sup E$ need not be a member of E. If $\sup E$ is in E then it is called the maximum element.

Completeness Property (or Least Upper Bound Property) of $\mathbb{R}$: Every non-empty set $E \subset \mathbb{R}$ that is bounded above has a least upper bound in $\mathbb{R}$.

1.14 Remark. $\mathbb{Q}$ does not have LUB property.

Proof. Let $E = \{p : p \in \mathbb{Q},\ p > 0,\ p^2 < 2\}$ is a non-empty subset of $\mathbb{Q}$ that is bounded above but it has no least upper bound in $\mathbb{Q}$. Assume $b \in \mathbb{Q}$ is a least upper bound of E. Since $p = 1 \in E$ and $p \leq b$ we have $1 \leq b$. We have two possibilities:

(i) $b \in E$

(ii) $b \notin E$

If $b \in E$, then $\exists q \in E$ such that $b < q$. Then b cannot be an upper bound. So $b \notin E$. Since $b \in \mathbb{Q}$ and $0 < b$, we have that $b^2 < 2$ is not true. By trichotomy law, we have either $b^2 = 2$ or $2 < b^2$. If $b \in \mathbb{Q}$, $b^2 = 2$ cannot be true. So $2 < b^2$.

Let $F = \{p : p \in \mathbb{Q},\ p > 0,\ 2 < p^2\}$. Then $b \in F$ and there is an element q in F such that $q < b$. Show q is an upper bound for E. Let $p \in E$ be an arbitrary element. Then $p > 0$ and $p^2 < 2$. Also $q \in F$ so $q > 0$ and $2 < q^2$.

$$p^2 < 2 \text{ and } 2 < q^2 \Rightarrow p^2 < q^2 \Rightarrow p < q$$

So q is bigger than every $p \in E$, i.e. q is an upper bound for E. Then $b \leq q$. Also $q < b$. Contradiction. $\square$

1.15 Remark. Analogously we have *greatest lower bound* (glb) or *infimum* (inf) and the Greatest Lower Bound Property.

1.16 Theorem.

(a) (Archimedian Property) For every $x, y \in \mathbb{R}$, $x > 0$, there is $n \in \mathbb{N}$ (depending on x, y) such that $nx > y$.

(b) ($\mathbb{Q}$ is dense in $\mathbb{R}$) For every $x, y \in \mathbb{R}$ with $x < y$, $\exists p \in \mathbb{Q}$ such that $x < p < y$.

Proof.

(a) Assume it is not true. So there are $x, y \in \mathbb{R}$ such that $x > 0$ for which there is no $n \in \mathbb{N}$ such that $nx > y$. So for all $n \in \mathbb{N}$ we have $nx \leq y$. Let $E = \{nx : n \in \mathbb{N}\} = \{x, 2x, 3x, \ldots\}$. Then y is an upper bound for E and $E \neq \emptyset$. So E has a least upper bound, say $\alpha \in \mathbb{R}$. Since $x > 0$, $\alpha - x < \alpha$. Then $\alpha - x$ is not an upper bound for E. So there is an element of E, say mx (where $m \in \mathbb{N}$) such that $\alpha - x < mx$. Then $\alpha < (m+1)x$. This element of E is bigger than $\alpha = \sup E$. Contradiction.

(b) Let $x, y \in \mathbb{R}$ be such that $x < y$. Then $y - x > 0$. By part (a), $\exists n \in \mathbb{N}$ such that $n(y-x) > 1$, i.e. $ny > 1 + nx$. In (a), replace y by nx and x by $1 > 0$. So $\exists m_1 \in \mathbb{N}$ such that $m_1 > nx$. Let $A = \{m : m \in \mathbb{Z},\ nx < m\}$. Then $A \neq \emptyset$ since $m_1 \in A$. Since A is non-empty set of integers which is bounded below, A has a smallest element m_0. Then $nx < m_0$ and $m_0 \in \mathbb{Z}$. $m_0 - 1 \notin A$. So we have $nx \geq m_0 - 1$. So $m_0 - 1 \leq nx < m_0$. Then $nx < m_0$, $m_0 \leq 1 + nx$ and $1 + nx < ny$. So $nx < m_0 < ny$. If we divide this by $n > 0$, we get $x < \frac{m_0}{n} < y$. $\frac{m_0}{n}$ is a rational number. $\square$

Uniqueness of Least Upper Bound: Assume $E \subset \mathbb{R}$ is non-empty and bounded above. Then E has only one least upper bound.

Proof. Assume b, b' are two least upper bounds for E.

(i) $b = \sup E$ and b' is an upper bound for $E \Rightarrow b \leq b'$

(ii) $b' = \sup E$ and b is an upper bound for $E \Rightarrow b' \leq b$

Then $b = b'$. $\square$

1.17 Fact. Let $E \subset \mathbb{R}$ be non-empty and bounded above and let $\alpha \in \mathbb{R}$. Then

$$\alpha = \sup E \Leftrightarrow \begin{array}{l} \text{(i) } \forall x \in E,\ \ x \leq \alpha \\ \text{and} \\ \text{(ii) Given any } \varepsilon > 0,\ \ \exists y \in E \text{ such that } \alpha - \varepsilon < y \end{array}$$

1.2 Extended Real Numbers

In the set of extended real numbers we consider the set $\mathbb{R} \cup \{-\infty, +\infty\}$. Preserve the order of $\mathbb{R}$ and $\forall x \in \mathbb{R}$, set $-\infty < +\infty$. This way every non-empty subset E of $\mathbb{R} \cup \{-\infty, +\infty\}$ has a least upper bound and greatest lower bound in $\mathbb{R} \cup \{-\infty, +\infty\}$. For example, if $E = \mathbb{N}$, then $\sup E = +\infty$. Also for $x \in \mathbb{R}$, we define the following

$$x + \infty = +\infty, \quad x + (-\infty) = -\infty, \quad \frac{x}{+\infty} = \frac{x}{-\infty} = 0$$

If $x > 0$, we define $x \cdot (+\infty) = +\infty$ and $x \cdot (-\infty) = -\infty$

If $x < 0$, we define $x \cdot (+\infty) = -\infty$ and $x \cdot (-\infty) = +\infty$

$0 \cdot (\mp\infty)$ is *undefined.*

1.3 The Complex Field

Let $\mathbb{C}$ denote the set of all ordered pairs (a,b) where $a,b \in \mathbb{R}$. We say

$$(a,b) = (c,d) \Leftrightarrow a = c \text{ and } b = d$$

Let $x = (a,b)$, $y = (c,d) \in \mathbb{C}$. We define

$$x + y = (a+c, b+d) \quad \text{and} \quad x \cdot y = (ac - bd, ad + bc)$$

Under these operations $\mathbb{C}$ becomes a field with $(0,0), (1,0)$ being the zero element and multiplicative unit.

Define $\phi : \mathbb{R} \to \mathbb{C}$ by $\phi(a) = (a,0)$. Then

$$\begin{aligned}
\phi(a+c) &= \phi(a+c,0) = (a,0) + (c,0) = \phi(a) + \phi(c) \\
\phi(ac) &= (ac,0) = (a,0)(c,0) = \phi(a) \cdot \phi(c)
\end{aligned}$$

ϕ is 1-1 (one-to-one), i.e. if $a \neq a'$ then $\phi(a) \neq \phi(a')$. This way we can identify $\mathbb{R}$ with the subset $\{(a,0) : a \in \mathbb{R}\}$ by means of ϕ.

Let $i = (0,1)$. Then $i^2 = (0,1) \cdot (0,1) = (-1,0) = \phi(-1)$. Also if $(a,b) \in \mathbb{C}$, then

$$\phi(a) + i\phi(b) = (a,0) + (0,1)(b,0) = (a,0) + (0,b) = (a,b)$$

If we identify $\phi(a)$ with a then we identify (a,b) with $a + ib$. So $\mathbb{C}$ is the set of all imaginary numbers in the form $a + ib$ where $a,b \in \mathbb{R}$ and $i^2 = -1$.

If $z = x + iy \in \mathbb{C}$, $x, y \in \mathbb{R}$ we define

$\overline{z} = x - iy$ (Conjugate of z)

$x = \text{Re } z$ (Real Part of z)

$y = \text{Im } z$ (Imaginary Part of z)

If $z, w \in \mathbb{C}$, we have

$$\overline{z+w} = \overline{z} + \overline{w}, \quad \overline{zw} = \overline{z} \cdot \overline{w}, \quad \text{Re } z = \frac{z + \overline{z}}{2}, \quad \text{Im } z = \frac{z - \overline{z}}{2i}$$

Since $z\overline{z} = x^2 + y^2 \geq 0$, we define the *modulus* of z as $|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}$.

1.18 Proposition. Let $z, w \in \mathbb{C}$. Then

(a) $z \neq 0 \Rightarrow |z| > 0$ and $z = 0 \Rightarrow |z| = 0$

(b) $|z| = |\overline{z}|$

(c) $|zw| = |z||w|$

(d) $|\mathrm{Re}\, z| \leq |z|$ (i.e. $|x| \leq \sqrt{x^2+y^2}$)

(e) $|z+w| \leq |z| + |w|$ (Triangle Inequality)

Proof of (e).

$$\begin{aligned} |z+w|^2 &= (z+w)(\overline{z+w}) = (z+w)(\overline{z}+\overline{w}) \\ &= z\overline{z} + \underbrace{z\overline{w} + w\overline{z}}_{2\mathrm{Re}(z\overline{w})} + w\overline{w} = |z|^2 + 2 \underbrace{\mathrm{Re}(z\overline{w})}_{\leq |z\overline{w}| = |z||w|} + |w|^2 \\ &\leq |z|^2 + 2|z||w| + |w|^2 = (|z|+|w|)^2 \end{aligned}$$

So $|z+w|^2 \leq (|z|+|w|)^2$. If we take positive square root of both sides, we get $|z+w| \leq |z|+|w|$. □

1.19 Theorem (Cauchy-Schwarz Inequality)**.** Let $a_j, b_j \in \mathbb{C}$ where $j = 1, 2, \ldots, n$. Then

$$\left| \sum_{j=1}^{n} a_j \overline{b_j} \right|^2 \leq \left(\sum_{j=1}^{n} |a_j|^2 \right) \left(\sum_{j=1}^{n} |b_j|^2 \right)$$

Proof. Let us define

$$A = \sum_{j=1}^{n} |a_j|^2 \quad , \quad B = \sum_{j=1}^{n} |b_j|^2 \quad , \quad C = \sum_{j=1}^{n} a_j \overline{b_j}$$

We have $B \geq 0$. If $B = 0$, then all $b_j = 0$. Then inequality becomes $0 \leq 0$. So assume $B > 0$. Then

$$\begin{aligned} 0 \leq \sum_{j=1}^{n} |Ba_j - Cb_j|^2 &= \sum_{j=1}^{n} (Ba_j - Cb_j) \left(B\overline{a_j} - \overline{C}\ \overline{b_j} \right) \\ &= \sum_{j=1}^{n} \left(B^2 |a_j|^2 - B\overline{C} a_j \overline{b_j} - CB b_j \overline{a_j} + |C|^2 |b_j|^2 \right) \\ &= B^2 A - \underbrace{B\overline{C}C}_{B|C|^2} - \underbrace{CB\overline{C}}_{|C|^2 B} + |C|^2 B \\ &= B \left(AB - |C|^2 \right) \end{aligned}$$

So $0 \leq B\left(AB - |C|^2\right)$ and since $B > 0$, we have $AB - |C|^2 \geq 0$. Then $|C|^2 \leq AB$. □

Replacing b_j by $\overline{b_j}$ and using $\overline{\overline{z}} = z$, $|\overline{z}| = |z|$ we get

$$\left|\sum_{j=1}^{n} a_j b_j\right|^2 \leq \left(\sum_{j=1}^{n} |a_j|^2\right)\left(\sum_{j=1}^{n} |b_j|^2\right)$$

If $a_j \geq 0$, $b_j \geq 0$ are real numbers then

$$\left(\sum_{j=1}^{n} a_j b_j\right)^2 \leq \left(\sum_{j=1}^{n} a_j^2\right)\left(\sum_{j=1}^{n} b_j^2\right)$$

2 Sets And Functions

2.1 General

Let X and Y be two non-empty sets. A function $f : X \to Y$ is a rule which assigns to each $x \in X$ a unique element $y = f(x)$ in Y.

2.1 Example. Given $x \in \mathbb{R}$, consider its decimal expansion in which there is no infinite chain of 9's. $x = \frac{1}{4}$ is represented as $0.25000\cdots$ instead of $0.24999\cdots$ Let $y = f(x)$ be the fortyninth digit after the decimal point. We have $f : \mathbb{R} \to \{0, 1, 2, \ldots, 9\}$.

2.2 Definition. Let $f : X \to Y$, $A \subset X$. We define the *image* of A under f as the set $f(A) = \{y \in Y : \exists x \in A \text{ such that } y = f(x)\}$. Let $B \subset Y$. We define the *inverse image* of B under f as the set $f^{-1}(B) = \{x \in X : f(x) \in B\}$.

2.3 Example. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. Then

$A = (1, 2]$ then $f(A) = (1, 4]$
$A = (-1, 3)$ then $f(A) = [0, 9)$
$B = [1, 4]$ then $f^{-1}(B) = [-2, -1] \cup [1, 2]$
$B = [-1, 4]$ then $f^{-1}(B) = [-2, 2]$
$B = [-2, -1]$ then $f^{-1}(B) = \emptyset$

2.4 Definition. Let $f : X \to Y$ be a function. Then we define

$$X\text{: Domain of } f \quad \text{and} \quad f(X)\text{: Range of } f$$

If $f(X) = Y$, then f is called a *surjection* or *onto*.

If $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$ (equivalently $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$) then f is called *one-to-one* (*1-1*) or an *injection*.

If f is both an injection and a surjection, then f is called a *bijection* or a *1-1 correspondence*.

Let $f : X \to Y$, $A \subset X$, $B \subset Y$. We have

(a) $f(f^{-1}(B)) \subset B$. $f(f^{-1}(B)) = B$ for all $B \Leftrightarrow f$ is onto

(b) $A \subset f(f^{-1}(A))$. $A = f(f^{-1}(A))$ for all $A \Leftrightarrow f$ is 1-1

(c) $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$. $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ for all A_1 and $A_2 \Leftrightarrow f$ is 1-1

(d) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$

Proof of (b).

$A \subset f^{-1}(f(A))$

Let $x \in A$ and let $y = f(x)$. Then $y \in f(A)$, i.e. $f(x) \in f(A)$. So $x \in f^{-1}(f(A))$ and $A \subset f^{-1}(f(A))$.

$A = f^{-1}(f(A))$ for all $A \subset X \Leftrightarrow f$ is 1-1

($\Leftarrow$): Assume f is 1-1. Show for all $A \subset X$, $A = f^{-1}(f(A))$, i.e. show $\underbrace{A \subset f^{-1}(f(A))}_{\text{always true}}$ and $f^{-1}(f(A)) \subset A$. So show $f^{-1}(f(A)) \subset A$. Let $x \in f^{-1}(f(A))$. Then $f(x) \in f(A)$. Then $\exists x' \in A$ such that $f(x) = f(x')$. Since f is 1-1, $x = x'$. So $x \in A$ and $f^{-1}(f(A)) \subset A$.

($\Rightarrow$): Assume $A = f^{-1}(f(A))$ for all $A \subset X$. Show f is 1-1. Assume f is not 1-1. Then there are $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$. Let $y = f(x_1) = f(x_2)$. Let $A = \{x_1\}$. Then $f(A) = \{y\}$ and $f^{-1}(f(A)) = \{x_1, x_2, \ldots\}$. Then $A \neq f^{-1}(f(A))$. Contradiction. $\square$

Let $\{A_i : i \in I\}$ be an arbitrary class of subsets of a set X indexed by a set I of subscripts. We define

$$\bigcup_{i \in I} A_i = \{x : x \in A_i \text{ for at least one } i \in I\}$$

$$\bigcap_{i \in I} A_i = \{x : x \in A_i \text{ for every } i \in I\}$$

If $I = \emptyset$, then we define $\bigcup_{i \in \emptyset} A_i = \emptyset$ and $\bigcap_{i \in \emptyset} A_i = X$. (If we require of an element that it belongs to each set in the class and if there are no sets in the class, then every element $x \in X$ satisfies this requirement.) If $A \subset X$ we define $A^C = \{x : x \notin A\}$ complement of A.

$$\left(\bigcup_{i \in I} A_i\right)^C = \bigcap_{i \in I} A_i^C \quad , \quad \left(\bigcap_{i \in I} A_i\right)^C = \bigcup_{i \in I} A_i^C \quad \text{(De Morgan's Laws)}$$

Let $f : X \to Y$, let $\{A_i : i \in I\}$, $\{B_j : j \in J\}$ be classes of subsets of X and Y.

$$f\left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} f(A_i) \quad , \quad f\left(\bigcap_{i \in I} A_i\right) \subset \bigcap_{i \in I} f(A_i)$$

$$f^{-1}\left(\bigcup_{j \in J} B_j\right) = \bigcup_{j \in J} f^{-1}(B_j) \quad , \quad f^{-1}\left(\bigcap_{j \in J} B_j\right) = \bigcap_{j \in J} f^{-1}(B_j)$$

For all $B \subset Y$ we have

(a) $f^{-1}(B^C) = (f^{-1}(B))^C$

(b) $f(A)^C \subset f(A^C)$ for all $A \subset X \Leftrightarrow f$ is onto

(c) $f(A^C) \subset f(A)^C$ for all $A \subset X \Leftrightarrow f$ is 1-1

Proof of (a). Let $x \in f^{-1}(B^C)$. Then $f(x) \in B^C$. Show $x \in (f^{-1}(B))^C$. Assume it is not true, i.e. $x \in f^{-1}(B) \Rightarrow f(x) \in B$. So $f(x) \in \underbrace{B^C \cap B}_{\emptyset}$. Contradiction. Thus, $x \in (f^{-1}(B))^C$. So, $f^{-1}(B^C) \subset (f^{-1}(B))^C$.
Let $x \in (f^{-1}(B))^C$. Show $x \in f^{-1}(B^C)$, i.e. $f(x) \in B^C$. Assume it is not true, i.e. $f(x) \in B$ so $x \in f^{-1}(B)$. Then, $x \in \underbrace{(f^{-1}(B))^C \cap f^{-1}(B)}_{\emptyset}$. Contradiction. So, $(f^{-1}(B))^C \subset f^{-1}(B^C)$. □

If $f : X \to Y$ is 1-1 onto, then $\forall y \in Y$ $\exists$ unique $x \in X$ such that $y = f(x)$. Send $Y \to X$. This way we get $f^{-1} : Y \to X$ (the inverse function of f) $f^{-1}(f(x)) = x$ $\forall x \in X$, $f(f^{-1}(y)) = y$ $\forall y \in Y$.

2.2 Countable And Uncountable Sets

Let $\mathscr{C}$ be any collection of sets. For $A, B \in \mathscr{C}$ we write $A \sim B$ (and say A and B are numerically equivalent) if there is a 1-1 correspondence $f : A \to B$. $\sim$ has the following properties

(i) $A \sim A$

(ii) $A \sim B \Rightarrow B \sim A$

(iii) $A \sim C \Rightarrow A \sim C$

2.5 Example. $A = \mathbb{N} = \{1, 2, 3, \ldots\}$, $B = 2\mathbb{N} = \{2, 4, 6, \ldots\}$. Then $A \sim B$ by $f : A \to B$, $f(n) = 2n$.

2.6 Definition. For $n = 1, 2, 3, \ldots$ let $J_n = \{1, 2, 3, \ldots, n\}$. Let $X \neq \emptyset$ be any set. We say

- X is *finite* if $\exists n \in \mathbb{N}$ such that $X \sim J_n$.
- X is *infinite* if it is not finite.
- X is *countable* (or *denumerable*) if $X \sim \mathbb{N}$.
- X is *uncountable* if X is not finite and not countable.

- X is *at most countable* if X is finite or countable.

2.7 Example. $\mathbb{N}$, $2\mathbb{N}$ are countable.

2.8 Example. $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is countable. Define $f : \mathbb{N} \to \mathbb{Z}$ as follows

$$\begin{array}{rllllll} \mathbb{N}: & 1, & 2, & 3, & 4, & 5, & \ldots \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots \\ \mathbb{Z}: & 0, & 1, & \text{-1}, & 2, & \text{-2}, & \ldots \end{array}$$

So we have

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

2.9 Example. $\mathbb{Q}^+ = \{q : q \in \mathbb{Q},\ q > 0\}$ is countable. Given $q \in \mathbb{Q}^+$, we have $m, n \in \mathbb{Z}$ $m > 0$, $n > 0$ such that $q = \frac{m}{n}$. List the elements of $\mathbb{Q}^+$ in

$$\begin{array}{c|ccccccccc} m\backslash n & 1 & & 2 & & 3 & & 4 & & \cdots \\ \hline 1 & \frac{1}{1} & & \frac{1}{2} & & \frac{1}{3} & & \frac{1}{4} & & \cdots \\ & & \swarrow & & \swarrow & & \swarrow & & & \\ 2 & \frac{2}{1} & & \frac{2}{2} & & \frac{2}{3} & & \frac{2}{4} & & \cdots \\ & & \swarrow & & \swarrow & & \swarrow & & & \\ 3 & \frac{3}{1} & & \frac{3}{2} & & \frac{3}{3} & & \frac{3}{4} & & \cdots \\ & & \swarrow & & \swarrow & & \swarrow & & & \\ 4 & \frac{4}{1} & & \frac{4}{2} & & \frac{4}{3} & & \frac{4}{4} & & \cdots \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots & & \ddots \end{array}$$

this order omitting the ones which are already listed before. Then we get the following sequence

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{1}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1} \cdots$$

Define $f : \mathbb{N} \to \mathbb{Q}^+$ as follows

$$f(1) = \frac{1}{1}\ ,\quad f(2) = \frac{1}{2}\ ,\quad f(3) = \frac{2}{1}\ ,\quad f(4) = \frac{1}{3}\quad \cdots$$

Suppose X is countable, so we have 1-1, onto function $f : \mathbb{N} \to X$. Let $f(n) = x_n$. Then we can write the elements of X as a sequence

$$X = \{x_1, x_2, x_3, \ldots\}$$

2.10 Example. $\mathbb{Q}$ is countable. (Similar to the proof of $\mathbb{Z}$ is countable.)

2.11 Proposition. Let I be a countable index set and assume for every $i \in I$, A_i is a countable set. Then $\bigcup_{i \in I} A_i$ is countable. (Countable union of countable sets is countable.)

2.12 Example. $X = [0, 1)$ is not countable. Every $x \in X$ has a binary (i.e. base 2) expansion $x = 0.x_1x_2x_3x_4 \cdots$ where each of $x_1, x_2, x_3, \ldots$ is 0 or 1. Assume $X = [0, 1)$ is countable. Then write its elements as a sequence. $X = \{y^1, y^2, y^3, \ldots\}$. Then write each of $y^1, y^2, y^3, \ldots$ in the binary expansion.

$$\begin{aligned} y^1 &= 0.y^1_1 y^1_2 y^1_3 \cdots \\ y^2 &= 0.y^2_1 y^2_2 y^2_3 \cdots \\ y^3 &= 0.y^3_1 y^3_2 y^3_3 \cdots \\ &\vdots \end{aligned}$$

Define

$$z_1 = \begin{cases} 0 & \text{if } y^1_1 = 1 \\ 1 & \text{if } y^1_1 = 0 \end{cases} \qquad z_2 = \begin{cases} 0 & \text{if } y^2_2 = 1 \\ 1 & \text{if } y^2_2 = 0 \end{cases} \qquad z_n = \begin{cases} 0 & \text{if } y^n_n = 1 \\ 1 & \text{if } y^n_n = 0 \end{cases}$$

Let $z = 0.z_1z_2z_3 \cdots \in [0, 1)$. But $z \neq y^1, z \neq y^2, \ldots$ Contradiction.

2.13 Example. $[0, 1]$, $(0, 1)$, $\mathbb{R}$, (a, b) are all uncountable. They are all numerically equivalent.

2.14 Example. $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ and $f(x) = \tan x$ gives $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \sim \mathbb{R}$.

2.15 Theorem (Cantor-Schröder-Bernstein)**.** Let X, Y be two non-empty sets. Assume there are 1-1 functions. $f : X \to Y$ and $g : Y \to X$. Then $X \sim Y$.

2.16 Example. Let $X = \{(x, y) : 0 < x < 1,\ 0 < y < 1\}$. Then $X \sim (0, 1)$.
Define $g : (0, 1) \to X$ by $g(x) = \left(x, \frac{1}{2}\right)$.
Define $f : X \to (0, 1)$ as follows: Given $(x, y) \in X$, write X and Y in their decimal expansion with no infinite chain of 9's.

$$x = 0.x_1 x_2 x_3 \cdots \quad \text{and} \quad y = 0.y_1 y_2 y_3 \cdots$$

Let

$$f(x, y) = 0.x_1 y_1 x_2 y_2 x_3 y_3 \cdots$$

Then f, g are 1-1 so by the above theorem $X \sim (0, 1)$.

3 Basic Topology

3.1 Metric Spaces

In $\mathbb{R}$, $\mathbb{R}^k = \{(x_1, \ldots, x_k) : x_1, \ldots, x_k \in \mathbb{R}\}$ we have the notion of distance.

In $\mathbb{R}$, $d(x, y) = |x - y|$

In $\mathbb{R}^k$, $x = (x_1, \ldots, x_k)$, $y = (y_1, \ldots, y_k)$. We have

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_k - y_k)^2}$$

3.1 Definition. Let $X \neq \emptyset$ be a set. Suppose we have a function

$$d : \underbrace{X \times X}_{\{(x,y):x,y\in X\}} \to \mathbb{R}$$

with the following properties

(i) $\forall x, y \in X$, $d(x, y) \geq 0 \quad d(x, y) = 0 \Leftrightarrow x = y$

(ii) $\forall x, y \in X$, $d(x, y) = d(y, x)$

(iii) $\forall x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle Inequality)

d is called a *metric* and the pair (X, d) is called a *metric space.*

Proof of (iii). $X = \mathbb{R}^k$, $d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_k - y_k)^2}$ then d satisfies (i) and (ii). For triangle inequality, let $x, y, z \in \mathbb{R}^k$ be given. Then we have

$$\begin{aligned}
d(x, y)^2 =&(x_1 - y_1)^2 + \cdots + (x_k - y_k)^2 \\
=&((x_1 - z_1) + (z_1 - y_1))^2 + \cdots + ((x_k - z_k) + (z_k - y_k))^2 \\
=&(x_1 - z_1)^2 + \cdots + (x_k - z_k)^2 + (z_1 - y_1)^2 + \cdots + (z_k - y_k)^2 \\
&+ 2[(x_1 - z_1)(z_1 - y_1) + \cdots + (x_k - z_k)(z_k - y_k)] \\
\leq&(d(x, z))^2 + (d(z, y))^2 \\
&+ 2\underbrace{|(x_1 - z_1)(z_1 - y_1) + \cdots + (x_k - z_k)(z_k - y_k)|}_{\substack{\leq((x_1-z_1)^2+\cdots+(x_k-z_k)^2)^{1/2}((z_1-y_1)^2+\cdots+(z_k-y_k)^2)^{1/2} \\ \text{(from Cauchy-Schwarz inequality)}}} \\
\leq&(d(x, z))^2 + (d(z, y))^2 + 2d(x, z)d(z, y) \\
\leq&(d(x, z) + d(z, y))^2
\end{aligned}$$

□

3.2 Example.

1) $X \neq \emptyset$ any set. Given any $x, y \in X$ let

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

(i) and (ii) are trivially true. Check (iii).

Proof of (iii). If $d(x,y) = 0$, then it is true as RHS ≥ 0. If $d(x,y) = 1$, then we cannot have RHS $= 0$. If RHS $= 0$ we have, $d(x,z) = 0$ and $d(z,y) = 0$. That is $x = z$ and $z = y$. So $x = y$. Then $d(x,y) = 0$. Contradiction. □

This metric is called the *discrete metric.*

2) Let $X = \mathbb{R}^k = \{x = (x_1, \ldots, x_k) : x_1, \cdots, x_k \in \mathbb{R}\}$

$$d_1(x,y) = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_k - y_k|$$

(i) and (ii) are trivially true. Check (iii).

Proof of (iii). Given $x, y, z \in \mathbb{R}^k$

$$\begin{aligned} d_1(x,y) &= |x_1 - y_1| + \cdots + |x_k - y_k| \\ &= |(x_1 - z_1) + (z_1 - y_1)| + \cdots + |(x_k - z_k) + (z_k - y_k)| \\ &\leq |x_1 - z_1| + |z_1 - y_1| + \cdots + |x_k - z_k| + |z_k - y_k| \\ &= d_1(x,z) + d_1(z,y) \end{aligned}$$

□

d_1 is called the ℓ_1 *metric* on $\mathbb{R}^k$.

3) $X = \mathbb{R}^k$

$$d_\infty(x,y) = \max\{|x_1 - y_1|, \ldots, |x_k - y_k|\}$$

(i) and (ii) are trivially true. Check (iii).

Proof of (iii). Given $x, y, z \in \mathbb{R}^k$

$$\begin{aligned} d_\infty(x,y) &= \max\{|x_1 - y_1|, \ldots, |x_k - y_k|\} \\ &= |x_i - y_i| \quad (1 \leq i \leq k) \\ &= |(x_i - z_i) + (z_i - y_i)| \leq \underbrace{|x_i - z_i|}_{\leq \max} + \underbrace{|z_i - y_i|}_{\leq \max} \\ &\leq d_\infty(x,z) + d_\infty(z,y) \end{aligned}$$

□

d_∞ is called the ℓ_∞ *metric* on $\mathbb{R}^k$.

4) Let S be any fixed non-empty set. A function $f : S \to \mathbb{R}$ is called *bounded* if $f(S)$ is a bounded subset of $\mathbb{R}$, i.e. there are two numbers A, B such that $\forall s \in S$, $A \leq f(s) \leq B$.

For example, $f : \mathbb{R} \to \mathbb{R}$, $f(s) = \arctan s$ is bounded, $f(s) = \sqrt{s^2+1}$ is unbounded.

Let $X = B(s) =$ all bounded functions $f : S \to \mathbb{R}$. Let $f, g \in B(S)$, then $f - g$ is also bounded. We define

$$d(f,g) = \sup\{|f(s) - g(s)| : s \in S\}$$

Geometrically, it is the supremum of the vertical distances between the two graphs (See Figure 1). (i) and (ii) are trivially true. Check (iii). We need the following: Let A, B be two non-empty subsets of $\mathbb{R}$ that are bounded above. Then, $\sup(A+B) \leq \sup A + \sup B$. $A + B = \{x + y : x \in A,\ y \in B\}$. Let $a = \sup A$ and $b = \sup B$. Let $z \in A + B$ be arbitrary. Then $\exists x \in A$, $y \in B$ such that $z = x + y$. Then we have $x + y \leq a + b$. Since $z = x + y$, $z \leq a + b$. So $a+b$ is an upper bound for the set $A+B$. Since supremum is the smallest upper bound, $\sup(A+B) \leq a+b$. In fact we have $\sup(A+B) = \sup A + \sup B$. Show $\sup(A+B) \geq \sup A + \sup B$. Given $\varepsilon > 0$, $\exists x \in A$ and $\exists y \in B$ such that $a - \varepsilon < x$ and $b - \varepsilon < y$. Then $a + b - 2\varepsilon < x + y$. Since $x + y \in A + B$, we have $x + y \leq \sup(A+B)$. Then $a + b - 2\varepsilon < \sup(A+B)$. Here a, b, $\sup(A+B)$ are fixed numbers, i.e. they do not depend on $\varepsilon > 0$. We have $a+b < \sup(A+B)+2\varepsilon$ true for every $\varepsilon > 0$. Then we have $a+b \leq \sup(A+B)$. If it is not true, then $a + b > \sup(A+B)$. Let $\delta = \frac{a+b-\sup(A+B)}{4}$ then $\delta > 0$. So we have

$$\begin{aligned}
a + b &< \sup(A+B) + 2\delta \\
&< \sup(A+B) + \frac{a+b-\sup(A+B)}{2} \\
\frac{a+b}{2} &< \frac{\sup(A+B)}{2}
\end{aligned}$$

Contradiction.

Proof of (iii). Given $f, g, h \in B(S)$, let

$$\begin{aligned}
C &= \{|f(s) - g(s)| : s \in S\} \\
A &= \{|f(s) - h(s)| : s \in S\} \\
B &= \{|h(s) - g(s)| : s \in S\}
\end{aligned}$$

Then $\sup C = d(f,g)$, $\sup A = d(f,h)$ and $supB = d(h,g)$. Let $x \in C$. Then $\exists s \in S$ such that $x = |f(s) - g(s)|$. Then

$$\begin{aligned} x &= |f(s) - g(s)| \\ &= |(f(s) - h(s)) + (h(s) - g(s))| \\ &\le \underbrace{|f(s) - h(s)|}_{\text{call } y} + \underbrace{|h(s) - g(s)|}_{\text{call } z} \\ &= y + z \in A + B \end{aligned}$$

So $\forall x \in C$, there is an element $u \in A + B$ such that $x \le u$. Then, $\sup C \le \sup(A + B) \le \sup A + \sup B$. That is, $d(f,g) \le d(f,h) + d(h,g)$. $\square$

3.3 Definition. Let (X,d) be a metric space, $p \in X$ and $r > 0$. Then

The *open ball* centered at p of radius r is defined as the set

$$B_r(p) = \{x \in X : d(x,p) < r\}$$

The *closed ball* centered at p of radius r is defined as the set

$$B_r[p] = \{x \in X : d(x,p) \le r\}$$

3.4 Example.

1) $X = \mathbb{R}^k$, $d_2(x,y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_k - y_k)^2}$. (See Figure 2).

2) $X \neq \emptyset$, d: discrete metric. Then

$$B_r(p) = \begin{cases} \{p\} & \text{if } r \le 1 \\ X & \text{if } r > 1 \end{cases} \qquad B_r[p] = \begin{cases} \{p\} & \text{if } r < 1 \\ X & \text{if } r \ge 1 \end{cases}$$

3) $X = \mathbb{R}^k$ with ℓ_1 metric d_1. Let $X = \mathbb{R}^2$, $p = (p_1, p_2)$, $x = (x_1, x_2)$. Then $|x_1 - p_1| + |x_2 - p_2| < r$. If $p_1 = p_2 = 0$ we have $|x_1| + |x_2| < r$. (See Figure 3).

4) $X = \mathbb{R}^k$ with ℓ_∞ metric d_∞. Let $X = \mathbb{R}^2$, $p = (0,0)$, $x = (x_1, x_2)$. Then $\max\{|x_1|, |x_2|\} < r$. (See Figure 4).

5) $X = B(S)$ with sup metric. Let $S = [a,b]$. Then $B_r(f)$ is the set of all functions g whose graph is in the shaded area. (See Figure 5).

3.5 Definition. Let (X, d) be a metric space and E a subset of X.

1) A point $p \in E$ is called an *interior point* of E if $\exists r > 0$ such that $B_r(p) \subset E$. (See Figure 6).

2) The set of all interior points of E is called the *interior* of E. It is denoted by $\operatorname{int} E$ or E°. We have $\operatorname{int} E \subset E$. (See Figure 7).

3) E is said to be *open* if $\operatorname{int} E = E$, i.e. if every point of E is an interior point of E, i.e. $\forall p \in E \; \exists r > 0$ such that $B_r(p) \subset E$. (See Figure 8).

3.6 Proposition. Every open ball $B_r(p)$ is an open set.

Proof. Let $q \in B_r(p)$. Show that $\exists s > 0$ such that $B_s(q) \subset B_r(p)$. Since $q \in B_r(p)$ we have $d(q, p) < r$. So $\underbrace{r - d(q, p)}_{\text{let this be } s} > 0$. Show $B_s(q) \subset B_r(p)$. Let $x \in B_s(q)$, i.e. $d(x, q) < s$. Then

$$\begin{aligned} d(x, p) &\le d(x, q) + d(q, p) \\ &< s + d(q, p) \\ &< r - d(q, p) + d(q, p) \\ &< r \end{aligned}$$

So $x \in B_r(p)$. $\square$

4) Let $p \in X$. A subset $N \subset X$ is called a *neighborhood* of p if $p \in \operatorname{int} N$. (See Figure 9).

   $B_r(p)$ is a neighborhood of p or $B_r(p)$ is a neighborhood of all of its points.

5) A point $p \in X$ is called a *limit point* (or *accumulation point* or *cluster point*) of E if every neighborhood N of p contains a point $q \in E$ with $q \neq p$. (See Figure 10).

6) A point $p \in E$ is called an *isolated point* of E if p is not a limit point of E, i.e. if there is a neighborhood N of p such that $N \cap E = \{p\}$.

3.7 Example. Let $X = \mathbb{R}$, $E = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$, $d(x, y) = |x - y|$. Isolated points of E are $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ E has only one limit point that is 0. Given any open ball $B_r(0) = (-r, r)$, find $n \in \mathbb{N}$ such that $\frac{1}{r} < n$. Then $x = \frac{1}{n} \in B_r(0) \cap E$ and $x \neq 0$.

7) E' is the set of all limit points of E. We define $\overline{E} = E \cup E'$. $\overline{E}$ is called the *closure* of E.

   We have $p \in \overline{E} \Leftrightarrow$ for every neighborhood N of p we have $N \cap E \neq \emptyset$.

8) E is *closed* if every limit point of E is an element of E, i.e. $E' \subset E$, i.e. $\overline{E} = E$. (See Figure 11).

9) E is *perfect* if E is closed and has no isolated points. (See Figure 12).

10) E is *bounded* if $\exists m > 0$ such that $\forall x, y \in E$ $d(x, y) \leq m$. (See Figure 13).

11) E is *dense* in X if $\overline{E} = X$, i.e. $\forall p \in X$ and for all neighborhood N of p we have $N \cap E \neq \emptyset$.

3.8 Example. Let $X = \mathbb{R}$, $E = \mathbb{Q}$. We have $\overline{\mathbb{Q}} = \mathbb{R}$. Given $p \in \mathbb{R}$ and given a neighborhood $B_r(p) = (p - r, p + r)$, find a rational number x such that $p - r < x < p + r$. So $x \in B_r(p) \cap \mathbb{Q}$.

3.9 Theorem. E is open $\Leftrightarrow$ E^C is closed.

Proof.

($\Rightarrow$): Let E be open. Show that every limit point p of E^C is an element of E^C. Assume it is not true. Then E^C has a limit point p_0 such that $p_0 \notin E^C$. Then $p_0 \in E$. Since E is open $\exists r > 0$ such that $B_r(p_0) \subset E$. Also since p_0 is a limit point of E^C, every neighborhood N of p_0 contains a point $q \in E^C$ such that $q \neq p_0$. Since $B_r(p_0)$ is also a neighborhood of p_0 we have that $B_r(p_0)$ contains a point $q \in E^C$ such that $q \neq p_0$. Then $q \in B_r(p_0) \subset E$ but also $q \in E^C$. Contradiction.

($\Leftarrow$): Let E^C be closed. Show that every point p in E is an interior point. Let $p \in E$ be arbitrary. Since $p \notin E^C$, p is not a limit point of E^C. Then p has a neighborhood N such that N does not contain any point $q \in E^C$ with $q \neq p$. Then $N \cap E^C = \emptyset$ so $N \subset E$. Since N is a neighborhood of p, $\exists r > o$ such that $B_r(p) \subset N$. So $B_r(p) \subset E$. $\square$

3.10 Theorem. p is a limit point of $E \Leftrightarrow$ every neighborhood N of p contains infinitely many points of E.

Proof.

($\Leftarrow$): Trivial.

($\Rightarrow$): Let p be a limit point of E and let N be an arbitrary neighborhood of p. Then $\exists r > 0$ such that $B_r(p) \subset N$.

$\exists q_1 \in B_r(p) \cap E$ such that $q_1 \neq p$. $d(q_1, p) > 0$. Let $r_1 = d(q_1, p) < r$.

$\exists q_2 \in B_{r_1}(p) \cap E$ such that $q_2 \neq p$. $d(q_2, p) < r_1 = d(q_1, p)$. Then $q_2 \neq q_1$. $d(q_2, p) > 0$. Let $r_2 = d(q_2, p) < r_1$.

Continue this way and get a sequence of points $q_1, q_2, q_3, \ldots, q_n, \ldots$ in E such that $q_i \neq q_j$ for $i + j$ and also each $q_i \neq p$ and $q_i \in B_r(p)$. $\square$

3.11 Corollary. If E is a finite set then E has no limit points.

3.12 Theorem. Let $E \subset X$. Then

(a) $\overline{E}$ is a closed set.

(b) $E = \overline{E} \Leftrightarrow E$ is a closed set.

(c) $\overline{E}$ is the smallest closed set that contains E, i.e. if F is any closed set such that $E \subset F$ then $\overline{E} \subset F$.

Proof.

(a) Show $(\overline{E})^C$ is an open set. Let $p \in (\overline{E})^C$. Then $\exists r > 0$ such that $B_r(p) \cap E = \emptyset$. This means $B_r(p) \subset E^C$. Show that actually $B_r(p) \subset (\overline{E})^C$. If not true, $\exists q \in B_r(p)$ such that $q \notin (\overline{E})^C$, i.e. $q \in \overline{E}$. Then for every neighborhood N of q we have $N \cap E \neq \emptyset$. This is also true for $N = B_r(p)$, i.e. $B_r(p) \cap E \neq \emptyset$. But $B_r(p) \cap E = \emptyset$. Contradiction. So $B_r(p) \subset (\overline{E})^C$.

(b) ($\Rightarrow$): $\overline{E}$ is closed by (a) so E is closed.

($\Leftarrow$): Let E be closed. Then E is contains all of its limit points, i.e. $E' \subset E$, $\overline{E} = E \cup E' \subset E$. Since $E \subset \overline{E}$ is always true, we have $\overline{E} = E$.

(c) Let E be given and F be a closed set such that $E \subset F$. Show $\overline{E} \subset F$. Let $p \in \overline{E} = E \cup E'$. If $p \in E$ then $p \in F$. If $p \in E'$ show $p \in F$. Given any neighborhood N of p, N contains infinitely many points of E so N contains infinitely many points of F, i.e. $p \in F' \subset F$. $\square$

Basic properties: Let (X, d) be a metric space.

1) The union of any collection of open sets is open.

2) The intersection of a *finite* number of open sets is open.

3) The intersection of any collection of closed sets is closed.

4) The union of a *finite* number of closed sets is closed.

5) E is open $\Leftrightarrow E^C$ is closed.

6) E is closed $\Leftrightarrow \overline{E} = E$.

7) $\overline{E}$ is the smallest closed set that contains E.

8) E is open $\Leftrightarrow E = \operatorname{int} E$.

9) $\operatorname{int} E$ is the largest open set that is contained in E.

3.13 Example. Intersection of infinitely many open sets may not be open. Let $X = \mathbb{R}$, $d(x,y) = |x-y|$. For $n = 1, 2, 3, \ldots$ let $E_n = \left(-\frac{1}{n}, \frac{n+1}{n}\right)$. All E_n's are open but $\bigcap_{n=1}^{\infty} E_n = [0,1]$ is not open.

3.14 Proposition. Let $\emptyset \neq E \subset \mathbb{R}$ be bounded above. Let $y = \sup E$. Then $y \in \overline{E}$.

Proof. Let N be an arbitrary neighborhood of y. Then $\exists r > 0$ such that $B_r(y) \subset N$. $y - r$ cannot be an upper bound for E. Then $\exists x \in E$ such that $y - r < x$. Also, $x \leq y < y + r$. So $y - r < x < y + r$, i.e. $x \in B_r(y) \Rightarrow x \in N$. So $x \in E \cap N$, i.e. $E \cap N \neq \emptyset$. □

3.2 Subspaces

3.15 Definition. Let (X,d) be a metric space and $Y \neq \emptyset$ be a subset of X. Then Y is a metric space in its own right with the same distance function. In this case we say (Y,d) is a *subspace* of (X,d).

3.16 Example. $X = \mathbb{R}^2$, $Y = \{(x,0) : x \in \mathbb{R}\}$. Let $E = \{(x,0) : 1 < x < 2\}$. Then $E \subset Y$ so $E \subset X$. As a subset of Y, E is an open set. As a subset of X, E is not an open set. Let $E \subset Y \subset X$. We say E is open (closed) relative to Y if E is open (closed) as a subset of the metric space (Y,d).

E is open relative to $Y \Leftrightarrow \forall p \in E\ \exists r > 0$ such that $\underbrace{B_r^Y(p)}_{B_r^X(p) \cap Y} \subset E$.

E is closed relative to $Y \Leftrightarrow Y \setminus E$ is open relative to Y.

3.17 Theorem. Let $E \subset Y \subset X$. Then

(a) E is open relative to Y $\Leftrightarrow$ there is an open set $F \subset X$ such that $E = F \cap Y$.

(b) E is closed relative to Y $\Leftrightarrow$ there is a closed set $F \subset X$ such that $E = F \cap Y$.

Proof.

(a) ($\Rightarrow$): Let E be open relative to Y. Then $\forall p \in E$ $\exists r_p > 0$ such that $B^X_{r_p}(p) \cap Y \subset E$. Let $F = \bigcup_{p \in E} B^X_{r_p}(p)$. Then F is an open set in X. Show $F \cap Y = E$.

$$F \cap Y = \left(\bigcup_{p \in E} B^X_{r_p}(p) \right) \cap Y = \bigcup_{p \in E} \underbrace{\left(B^X_{r_p}(p) \cap Y \right)}_{\subset E \text{ for all } p} \subset E$$

Conversely, let $p_0 \in E$. Then $p_0 \in Y$ since $E \subset Y$. Then $p_0 \in B^X_{r_{p_0}} \subset F$. So $p_0 \in Y \cap F$.

($\Leftarrow$): Assume $E = F \cap Y$ where $F \subset X$ is open in X. Let $p \in E$. Then $p \in Y$ and $p \in F$. Since F is open in X, $\exists r > 0$ such that $B^X_r(p) \subset F$. Then $B^X_r(p) \cap Y \subset F \cap Y = E$.

(b) ($\Rightarrow$): Let $E \subset Y$ be closed relative to Y. Then $Y \setminus E$ is open relative to Y. So there exists an open set $A \subset X$ such that $Y \setminus E = A \cap Y$. Then $E = Y \setminus (Y \setminus E) = Y \setminus (A \cap Y) = (A \cap Y)^C \cap Y = (A^C \cup Y^C) \cap Y = (A^C \cap Y) \cup \underbrace{(Y^C \cap Y)}_{\emptyset} = A^C \cap Y$. A^C is closed in X and we may call it F.

($\Leftarrow$): Assume $E = F \cap Y$ where $F \subset X$ is a closed set. Then $Y \setminus E = Y \setminus (F \cap Y) = Y \cap (F \cap Y)^C = Y \cap (F^C \cup Y^C) = Y \cap \underbrace{F^C}_{\text{open in } X}$ is open relative to Y. $\square$

3.3 Compact Sets

3.18 Definition. Let (X, d) be a metric space and E be a non-empty subset of X. An open cover of X is a collection of $\{G_i : i \in I\}$ of open subsets G_i of X such that $E \subset \bigcup_{i \in I} G_i$.

3.19 Example. $X = \mathbb{R}$, $d(x, y) = |x - y|$, $E = (0, 1)$. For every $x \in E$ let $G_x = (-1, x)$. $\bigcup_{x \in E} G_x = (-1, 1) \supset E$. So $\{G_x : x \in E\}$ is an open cover of E.

3.20 Example. $X = \mathbb{R}^2$ with d_2 and $E = B_1(\underline{0})$. For $n \in \mathbb{N}$ let $G_n = B_{\frac{n}{n+1}}(\underline{0})$. $\bigcup_{n=1}^{\infty} G_n = E$. So $\{G_n : n \in \mathbb{N}\}$ is an open cover of E.

3.21 Definition. A subset K of a metric space (X, d) is said to be *compact* if every open cover of K contains a finite subcover, i.e. given any open cover $\{G_i : i \in I\}$ of K, we have that $\exists i_1, \ldots, i_n \in I$ such that $K \subset G_{i_1} \cup \cdots \cup G_{i_n}$.

3.22 Example. $X = \mathbb{R}$, $d(x, y) = |x - y|$. $E = (0, 1)$ is not compact. Take $I = E = (0, 1)$. For $x \in I$, $G_x = (-1, x)$. $\bigcup_{x \in I} G_x = (-1, 1) \supset E$. So, $\{G_x : x \in I\}$ is an open cover for E. This open cover does not have any finite subcover. Assume it is not true, so assume $x_1, \ldots, x_n \in E$ such that $G_{x_1} \cup \cdots \cup G_{x_n} \supset E$. Let $x_k = \max\{x_1, \ldots, x_n\}$. Then $(0, 1) \subset (-1, x_k)$ but $x_k \in E = (0, 1)$, i.e. $x_k < 1$. Let $x = \frac{x_k + 1}{2}$. Then $x \in E$ but $x \notin (-1, x_k)$.

3.23 Theorem. Let $K \subset Y \subset X$. Then K is compact relative to $Y \Leftrightarrow K$ is compact relative to Y.

Proof.

($\Rightarrow$): Assume K is compact relative to Y. Let $\{G_i : i \in I\}$ be any collection of sets open relative to X such that $\bigcup_{i \in I} G_i \supset K$. Then $\{Y \cap G_i : i \in I\}$ is a collection of sets open relative to Y and $K = K \cap Y \subset \left(\bigcup_{i \in I} G_i\right) \cap Y = \bigcup_{i \in I}(Y \cap G_i)$. So $\{Y \cap G_i : i \in I\}$ is an open (relative to Y) cover of K. Since K is compact relative to Y, $\exists i_1, \ldots, i_n \in I$ such that $K \subset (Y \cap G_{i_1}) \cup \cdots \cup (Y \cap G_{i_n})$. Then $K \subset G_{i_1} \cup \cdots \cup G_{i_n}$.

($\Leftarrow$): Let K be compact relative to X. Let $\{G_i : i \in I\}$ be an arbitrary collection of sets open relative to Y such that $K \subset \bigcup_{i \in I} G_i$. Then we have $G_i = Y \cap E_i$ for some open subset E_i of X. Then $K \subset \bigcup_{i \in I}(Y \cap E_i) \subset \bigcup_{i \in I} E_i$. So $\{E_i : i \in I\}$ is an open cover of K in X. Since K is compact relative to X, $\exists i_1, \ldots, i_n$ such that $K \subset E_{i_1} \cup \cdots \cup E_{i_n}$. Then $K = K \cap Y \subset (E_{i_1} \cap Y) \cup \cdots \cup (E_{i_n} \cap Y) = G_{i_1} \cup \cdots \cup G_{i_n}$. $\square$

3.24 Theorem. If $K \subset X$ is compact then K is closed.

Proof. We will show K^C is open. Let $p \in K^C$. Show $\exists r > 0$ such that $B_r(p) \subset K^C$ $\forall q \in K$ (since $q \neq p$). $d(q,p) > 0$. Let

$$r_q = \frac{d(q,p)}{2} \qquad V_q = B_{r_q}(p) \qquad W_q = B_{r_q}(q)$$

Then $V_q \cap W_q = \emptyset$. Find V_q and W_q $\forall q \in K$ $K \subset \bigcup_{q \in K} W_q$. The collection $\{W_q : q \in K\}$ is an open cover of K. Since K is compact, $\exists q_1, \dots, q_n \in K$ such that $K \subset W_{q_1} \cup \cdots \cup W_{q_n}$. Let $V = V_{q_1} \cap \cdots \cap V_{q_n}$. Then V is an open set and $p \in V$. If $r_{q_k} = \min\{r_{q_1}, \dots, r_{q_n}\}$ then $V = B_{r_{q_k}}(p)$. Show $V \subset K^C$. If it is not true, then $\exists z \in V$ but $z \notin K^C$, i.e. $z \in K$. Then $z \in W_{q_i}$ for some $i \in \{1, \dots, n\}$. $z \in V \subset V_{q_i}$ (the same i). So $z \in W_{q_i} \cap V_{q_i} = \emptyset$. Contradiction. $\square$

3.25 Theorem. Closed subsets of compact sets are compact.

Proof. Let $F \subset K \subset X$ where K is compact and F is closed. (relative to X and relative to K are the same since K is closed.) Let $\{G_i : i \in I\}$ be a set open in X such that $F \subset \bigcup_{i \in I} G_i$. Then $\{G_i : i \in I\} \cup \{F^C\}$ is an open cover of K. Since K is compact, $\exists i_1, \dots, i_n \in I$ such that $K \subset G_{i_1} \cup \cdots \cup G_{i_n} \cup F^C$. Then $F = F \cap K \subset (G_{i_1} \cap F) \cup \cdots \cup (G_{i_n} \cap F) \cup \underbrace{(F^C \cap F)}_{\emptyset} \subset G_{i_1} \cup \cdots \cup G_{i_n}$. $\square$

3.26 Corollary. Let $F, K \subset X$ where K is compact and F is closed. Then $F \cap K$ is compact.

3.27 Theorem. Let $\{K_i : i \in I\}$ be a collection of compact subsets of X such that the intersection of every finite subcollection of $\{K_i : i \in I\}$ is non-empty. $\bigcap_{i \in I} K_i \neq \emptyset$.

Proof. Assume the contrary. $\bigcap_{i \in I} K_i = \emptyset$. Fix one of these sets, K_{i_0}. Then $K_{i_0} \cap \left(\bigcap_{i \neq i_0} K_i\right) = \emptyset$. Then $K_{i_0} \subset \left(\bigcap_{i \neq i_0} K_i\right)^C$, i.e. $K_{i_0} \subset \bigcup_{i \neq i_0} K_i^C$. So $\{K_i^C : i \neq i_0\}$ is an open cover of K_{i_0}. Since K_{i_0} is compact, $\exists i_1, \dots, i_n$ such that $K_{i_0} \subset \underbrace{K_{i_1}^C \cup \cdots \cup K_{i_n}^C}_{(K_{i_1} \cap \cdots \cap K_{i_n})^C}$. But $K_{i_0} \cap K_{i_1} \cap \cdots \cap K_{i_n} = \emptyset$. This contradicts the hypothesis. $\square$

3.28 Corollary. Let $\{K_n : n \in \mathbb{N}\}$ be a sequence of non-empty compact sets such that $K_1 \supset K_2 \supset K_3 \supset \cdots$ Then $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$.

3.29 Example. $X = \mathbb{R}$, $d(x,y) = |x-y|$, $E_n = [n, +\infty) = \{x : x \in \mathbb{R},\ x \geq n\}$, $n = 1, 2, 3, \ldots$ Then E_n's are closed, $E_n \neq \emptyset$ and $E_1 \supset E_2 \supset E_3 \supset \cdots$ We have $\bigcap_{n=1}^{\infty} E_n = \emptyset$.

3.30 Theorem (Nested Intervals)**.** Let $\{I_n : n = 1, 2, \ldots\}$ be a sequence of non-empty, closed, bounded intervals in $\mathbb{R}$ such that $I_1 \supset I_2 \supset I_3 \supset \cdots$ Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. Let $I_n = [a_n, b_n]$ and $a_n \leq b_n$. That is, $a_1 \leq a_2 \leq a_3 \leq \cdots$ and $\cdots \leq b_3 \leq b_2 \leq b_1$. We also have that $\forall n\ \forall k\ a_n \leq b_k$. Given $n, k \in \mathbb{N}$

(i) If $n = k$ then $a_n \leq b_n = b_k$.

(ii) If $n > k$ then $I_n \subset I_k$. Then $a_n \in I_n \subset I_k \Rightarrow a_k \leq a_n \leq b_k$.

(iii) If $n < k$ then $I_k \subset I_n$. Then $b_k \in I_k \subset I_n \Rightarrow a_n \leq b_k \leq b_n$.

Let $E = \{a_1, a_2, a_3, \ldots\}$. Then $E \neq \emptyset$ and E is bounded above. Let $x = \sup E$. Then $a_n \leq x$ for all n.

Claim: $x \leq b_n\ \forall n$.

Assume it is not true. Then $\exists n$ such that $b_{n_0} < X$. Then b_{n_0} cannot be an upper bound for E. So there is an element $a_{k_0} \in E$ such that $a_{k_0} > b_{n_0}$. Contradiction. So $\forall n$ we have $a_n \leq x \leq b_n$, i.e. $x \in I_n$. So $x \in \bigcap_{n=1}^{\infty} I_n$. $\square$

3.31 Remark. If also $\lim_{n\to\infty}(b_n - a_n) = 0$ then $\bigcap_{n=1}^{\infty} I_n$ consists of only one point.

3.32 Definition. Let $a_1 \leq b_1, \ldots, a_k \leq b_k$ be real numbers. Then the set of all points $p = (x_1, \ldots, x_k) \in \mathbb{R}^k$ such that $a_1 \leq x_1 \leq b_1, \ldots, a_k \leq x_k \leq b_k$ is called a k*-cell* in $\mathbb{R}^k$. (See Figure 14).

3.33 Theorem. Let k be fixed. Let $\{I_n\}$ be a sequence of non-empty k-cells such that $I_1 \supset I_2 \supset I_3 \supset \cdots$ Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

3.34 Theorem. Let K be a compact subset of a metric space (X, d). Then K is bounded.

Proof. Fix a point $p_0 \in K$. $X = \bigcup_{n=1}^{\infty} B_n(p_0)$ and $K \subset \bigcup_{n=1}^{\infty} B_n(p_0)$. So $\exists n_1, \ldots n_r \in \mathbb{N}$ such that $K \subset B_{n_1}(p_0) \cup \cdots \cup B_{n_r}(p_0)$. Let $n_0 = \max\{n_1, \ldots, n_r\}$. Let $p, q \in K$, then $p \in B_{n_i}(p_0)$ and $q \in B_{n_j}(p_0)$ where n_i, n_j are one of $n_1, \ldots, n_r$. We have $d(p,q) \leq d(p, p_0) + d(p_0, q) \leq n_i + n_j \leq 2n_0$. $\square$

3.35 Theorem (Heine-Borel)**.** A subset K of $\mathbb{R}^k$ is compact $\Leftrightarrow$ K is closed and bounded.

Proof.

($\Rightarrow$): True in all metric spaces.

($\Leftarrow$): Let $K \subset \mathbb{R}^k$ be closed and bounded. Since K is bounded, there is a k-cell I such that $K \subset I$. I is compact, so K being a closed subset of the compact set I is compact. □

3.36 Theorem. Every k-cell is a compact set in $\mathbb{R}^k$.

Proof. Let $E \subset \mathbb{R}^k$ be a k-cell. Then there are real numbers $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$ such that $a_1 \leq b_1, a_2 \leq b_2, \ldots, a_k \leq b_k$ and

$$E = \{p = (x_1, \ldots, x_k) \in \mathbb{R}^k : a_1 \leq x_1 \leq b_1, \ldots, a_k \leq x_k \leq b_k\}$$

If $a_1 = b_1, a_2 = b_2, \ldots, a_k = b_k$ then E consists of one point which is compact. So assume there is at least one j, $1 \leq j \leq k$, such that $a_j < b_j$. Let

$$\delta = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + \cdots + (b_n - a_n)^2}$$

Then $\delta > 0$. Assume E is not compact. So there is an open cover $\{G_\alpha : \alpha \in A\}$ such that no finite subcollection of G_α's covers E. Let

$$c_i = \frac{a_i + b_i}{2}$$

We divide each side of E into two parts and this way we divide E into 2^k subcells. Call them $Q_1, Q_2, \ldots, Q_{2^k}$. Then at least one of these Q_j's cannot be covered by finitely many sets, G_α's. Call this Q_j E_1. For all $p, q \in E$ we have $d_2(p, q) \leq \delta$ and for all $p, q \in E_1$, $d_2(p, q) \leq \frac{\delta}{2}$. Next divide E_1 into 2^k subcells by halving each side and continue this way. This way we obtain a sequence $\{E_n\}$ of k-cells such that

(a) $E \supset E_1 \supset E_2 \supset E_3 \supset \cdots$

(b) E_n cannot be covered by any finite subcollection of $\{G_\alpha : \alpha \in A\}$

(c) For all $p, q \in E_n$, $d_2(p, q) \leq \frac{\delta}{2^n}$

By (a), $\bigcap_{n=1}^{\infty} E_n \neq \emptyset$. Let $p^* \in \bigcap_{n=1}^{\infty} E_n$. Then $p^* \in E$. Since $\{G_\alpha : \alpha \in A\}$ is an open cover of E, there is an $\alpha_0 \in A$ such that $p^* \in G_{\alpha_0}$. Since G_{α_0} is open, there is $r > 0$ such that $B_r(p^*) \subset G_{\alpha_0}$. Find a natural number n_0 such that $\frac{\delta}{r} < 2^{n_0}$, i.e. $\frac{\delta}{2^{n_0}} < r$. Now we show that $E_{n_0} \subset G_{\alpha_0}$. $p^* \in E_{n_0}$. Let $p \in E_{n_0}$ be an arbitrary point. By (c), $d_2(p, p^*) \leq \frac{\delta}{2^{n_0}}$. Also, $\frac{\delta}{2^{n_0}} < r$. So $d_2(p, p^*) < r$. So $p \in B_r(p^*) \subset G_{\alpha_0}$. Thus, $p \in E_{n_0} \Rightarrow p \in G_{\alpha_0}$. So $E_{n_0} \subset G_{\alpha_0}$. This means E_{n_0} can be covered by finitely many sets from $\{G_\alpha : \alpha \in A\}$ (indeed just by one set). This contradicts (b). $\square$

3.37 Theorem. Let (X, d) be a metric space and $K \subset X$. Then K is compact $\Leftrightarrow$ every infinite subset of K has a limit point in K.

Proof.

($\Rightarrow$): Let K be compact. Assume claim is not true. Then there is an infinite subset $A \subset K$ such that A has no limit point in K. So, given any point $p \in K$, p is not a limit point of A. So there is $r_p > 0$ such that $B_{r_p}(p)$ contains no point of A different from p. The collection of $\{B_{r_p}(p) : p \in K\}$ is an open cover of K. Since K is compact, there are $p_1, p_2, \ldots, p_n \in K$ such that $K \subset B_{r_{p_1}}(p_1) \cup B_{r_{p_2}}(p_2) \cup \cdots \cup B_{r_{p_n}}(p_n)$. Since $A \subset K$ and A is an infinite set, one of the open balls on the right hand side must contain infinitely many points of A. Contradiction.

($\Leftarrow$): Omitted. $\square$

3.38 Theorem (Bolzano-Weierstrass)**.** Every infinite, bounded subset E of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

Proof. Since E is bounded, there is a k-cell I such that $E \subset I$. Then since I is compact, the infinite subset E of I has a limit point $p \in I \subset \mathbb{R}^k$. $\square$

3.4 The Cantor Set

Let us define

$$
\begin{aligned}
E_0 &= [0, 1] \\
E_1 &= \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \\
E_2 &= \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right] \\
&\vdots
\end{aligned}
$$

Continue this way by removing the open middle thirds of the remaining intervals. This way we get a sequence $E_1 \supset E_2 \supset E_3 \supset \cdots \supset E_n \supset \cdots$ such that E_n is the union of 2^n disjoint closed intervals of length $\frac{1}{3^n}$. We define

$$\mathcal{C} = \bigcap_{n=1}^{\infty} E_n$$

which is called the *Cantor Set.*

Properties of $\mathcal{C}$

(1) $\mathcal{C}$ is compact

(2) $\mathcal{C} \neq \emptyset$

(3) $\operatorname{int} \mathcal{C} = \emptyset$

(4) $\mathcal{C}$ is perfect

(5) $\mathcal{C}$ is uncountable

Proof of (3). Assume $\operatorname{int} \mathcal{C} \neq \emptyset$. Then there is a non-empty open interval $(\alpha, \beta) \subset \mathcal{C}$ and $\mathcal{C}$ does not contain intervals of the form $\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right)$. Since they are removed in the process of construction. Assume $\mathcal{C}$ contains (α, β) where $\alpha < \beta$. Let $a > 0$ be a constant which will be determined later. Choose $m \in \mathbb{N}$ such that $\frac{a}{\beta-\alpha} < 3^m$, i.e. $\frac{1}{3^m} < \frac{\beta-\alpha}{a}$. Let k be the smallest integer such that $\alpha < \frac{3k+1}{3^m}$, i.e. $\frac{\alpha 3^m - 1}{3} < k$. Then $k - 1 \leq \frac{\alpha 3^m - 1}{3}$. Show $\frac{3k+2}{3^m} < \beta$.

$$k - 1 \leq \frac{\alpha 3^m - 1}{3} \Rightarrow k \leq \frac{\alpha 3^m - 1}{3} + 1$$

Show

$$\begin{aligned}
\frac{\alpha 3^m - 1}{3} + 1 &< \frac{3^m \beta - 2}{3} \\
1 &< \frac{3^m \beta - 2}{3} - \frac{3^m \alpha - 1}{3} \\
1 &< \frac{3^m(\beta - \alpha) - 1}{3}
\end{aligned}$$

Now we have

$$\frac{3^m(\beta - \alpha) - 1}{3} > \frac{3^m a \frac{1}{3^m} - 1}{3} = \frac{a - 1}{3}$$

So let $\frac{a-1}{3} \geq 1$, i.e. $a \geq 4$. Choose $a = 4$. This way we have that

$$\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right) \subset (\alpha, \beta) \Rightarrow \left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right) \subset \mathcal{C}$$

Contradiction. $\square$

3.5 Connected Sets

3.39 Definition. Let (X, d) be a metric space and $A, B \subset X$. We say A and B are *separated* if $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$.

3.40 Example. $X = \mathbb{R}$, $A = \mathbb{Q}$, $B = \mathbb{R} \setminus \mathbb{Q}$. We have $\overline{A} \cap B = \mathbb{R} \cap (\mathbb{R} \setminus \mathbb{Q}) \neq \emptyset$. Then $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$ in $\mathbb{R}$ are not separated.

3.41 Definition. A subset $E \subset X$ is said to be *disconnected* if there are two non-empty separated sets A, B such that $E = A \cup B$.

A subset $E \subset X$ is said to be *connected* if it is not disconnected, i.e. there are no non-empty separated sets A, B such that $E = A \cup B$.

3.42 Theorem. A non-empty subset $E \subset \mathbb{R}$ is connected $\Leftrightarrow$ E is an interval.

Proof. An interval is defined as follows: Whenever $x < z$ and $x, z \in E$ for all y with $x < y < z$ we have $y \in E$.

($\Rightarrow$): Let $E \subset \mathbb{R}$ be connected. Assume E is not an interval. So there are two points $x, z \in E$ with $x < z$, there is y with $x < y < z$ and $y \notin E$. Let $A = E \cap (-\infty, y)$, $B = E \cap (y, +\infty)$. $x \in A$, $z \in B$. So $A \neq \emptyset$, $B \neq \emptyset$. Show $\overline{A} \cap B = \emptyset$, $A \cap \overline{B} = \emptyset$. $A \subset (-\infty, y) \Rightarrow \overline{A} \subset (-\infty, y]$. So $\overline{A} \cap B \subset (-\infty, y) \cap (y, +\infty) = \emptyset$. Similarly we have $A \cap \overline{B} = \emptyset$.

$$\begin{aligned} A \cup B &= (E \cap (-\infty, y)) \cup (E \cap (y, +\infty)) \\ &= E \cap \underbrace{((-\infty, y) \cup (y, +\infty))}_{\mathbb{R} \setminus \{y\}} = E \quad \text{since } y \notin E \end{aligned}$$

($\Leftarrow$): Omitted. $\square$

4 Sequences And Series

4.1 Sequences

Let (X, d) be a metric space. A sequence in X is a function $f : \mathbb{N} \to X$. If $p_n = f(n)$, we denote this sequence by (p_n) or $\{p_n\}$.

4.1 Example. $X = \mathbb{R}^2$, $p_n = \left(\frac{1-n}{n}, \frac{(-1)^n}{n}\right)$ where $n = 1, 2, 3, \ldots$ Then

$$\begin{aligned}
p_1 &= (0, -1) \\
p_2 &= \left(\frac{-1}{2}, \frac{1}{2}\right) \\
p_3 &= \left(\frac{-2}{3}, \frac{-1}{3}\right) \\
&\vdots
\end{aligned}$$

4.2 Definition. We say the sequence $\{p_n\}$ converges to $p \in X$ if for every $\varepsilon > 0$ there is a natural number n_0 (depending on $\varepsilon > 0$ in general) such that for all $n \in \mathbb{N}$ with $n \geq n_0$ we have $d(p_n, p) < \varepsilon$, i.e. $p_n \in B_\varepsilon(p)$. We write $p_n \to p$ or $\lim_{n\to\infty} p_n = p$.

$p_n \to p \Leftrightarrow$ every neighborhood of p contains all but finitely many terms p_n.

If $\{p_n\}$ does not converge to any $p \in X$, we say $\{p_n\}$ is divergent.

4.3 Example. $X = \mathbb{R}^2$, $p_n = \left(\frac{1-n}{n}, \frac{(-1)^n}{n}\right)$. $p = (-1, 0)$. Show $p_n \to p$. Let $\varepsilon > 0$ be given. Let n_0 be any natural number such that $\frac{\sqrt{2}}{\varepsilon} < n_0$. Let n be any natural number such that $n_0 \leq n$.

$$\begin{aligned}
d_2(p_n, p) &= \sqrt{\left(\frac{1-n}{n} - (-1)\right)^2 + \left(\frac{(-1)^n}{n} - 0\right)^2} \\
&= \sqrt{\left(\frac{1}{n}\right)^2 + \left(\frac{(-1)^n}{n}\right)^2} \\
&= \sqrt{\frac{2}{n^2}} = \frac{\sqrt{2}}{n} \\
&\leq \frac{\sqrt{2}}{n_0} < \varepsilon
\end{aligned}$$

4.4 Remark. $\{x_n\} = \left\{\frac{1}{n}\right\}$ converges to $x = 0$ in $(\mathbb{R}, |\cdot|)$, but it is divergent in $((0,2), |\cdot|)$.

4.5 Definition. We say that the sequence $\{p_n\}$ is bounded if the set $E = \{p_1, p_2, p_3, \ldots\}$ is a bounded subset of X, i.e. there is a constant $M > 0$ such that for all $p_i, p_j \in E$ we have $d(p_i, p_j) \leq M$.

4.6 Theorem.

(a) Let $\{p_n\}$ be a sequence such that $p_n \to p$ and $p_n \to p'$. Then $p = p'$.

(b) If $\{p_n\}$ is convergent then $\{p_n\}$ is bounded.

(c) Let $E \neq \emptyset$ be subset of X. Then $p \in \overline{E} \Leftrightarrow$ there is a sequence $\{p_n\}$ contained in E such that $p_n \to p$.

Proof.

(a) Let $p_n \to p$ and $p_n \to p'$. Assume $p \neq p'$. Then $d(p, p') > 0$. Let $\varepsilon_0 = \frac{d(p,p')}{3}$ then $\varepsilon_0 > 0$. We have $p_n \to p$ so there is $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$ we have $d(p_n, p) < \varepsilon_0$. We also have $p_n \to p'$ so there is $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$ we have $d(p_n, p') < \varepsilon_0$. Let $n_0 = \max\{n_1, n_2\}$. Then $n_0 \geq n_1 \Rightarrow d(p_{n_0}, p) < \varepsilon_0$ and $n_0 \geq n_2 \Rightarrow d(p_{n_0}, p') < \varepsilon_0$. Then $3\varepsilon_0 = d(p, p') \leq d(p, p_{n_0}) + d(p_{n_0}, p') < \varepsilon_0 + \varepsilon_0 = 2\varepsilon_0$. So $3\varepsilon_0 < 2\varepsilon_0$. Since $\varepsilon_0 > 0$, this cannot be true.

(b) Let $p_n \to p$. For $\varepsilon = 1 > 0$, there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $d(p_n, p) < 1$. If $i, j \geq n_0$, then $d(p_i, p_j) \leq d(p_i, p) + d(p, p_j) < 1 + 1 = 2$. Let $K = \max\{1, d(p_1, p), \ldots, d(p_{n_0 - 1}, p)\}$. Then for all $n \in \mathbb{N}$ we have $d(p_n, p) \leq K$. For any $i, j \in \mathbb{N}$, $d(p_i, p_j) \leq d(p_i, p) + d(p, p_j) \leq 2K$.

(c) ($\Leftarrow$): $\{p_n\}$ in E such that $p_n \to p$. Show $p \in \overline{E}$. Given $r > 0$ we have $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $d(p_n, p) < r$. So $p_{n_0} \in B_r(p) \cap E$. So $B_r(p) \cap E \neq \emptyset$. So $p \in \overline{E}$.

($\Rightarrow$): Let $p \in \overline{E}$. Then for every $r > 0$, $B_r(p) \cap E \neq \emptyset$.

$$\begin{aligned}
&\text{For } r = 1, \text{ find } p_1 \in B_1(p) \cap E \\
&\text{For } r = \frac{1}{2}, \text{ find } p_2 \in B_{\frac{1}{2}}(p) \cap E \\
&\qquad \vdots \\
&\text{For } r = \frac{1}{n}, \text{ find } p_n \in B_{\frac{1}{n}}(p) \cap E
\end{aligned}$$

Then $\{p_n\}$ is a sequence in E. Given $\varepsilon > 0$ let n_0 be such that $\frac{1}{\varepsilon} < n_0$. Then for all $n \geq n_0$ we have $d(p_n, p) < \frac{1}{n} \leq \frac{1}{n_0} < \varepsilon$. So $p_n \to p$. $\square$

4.7 Theorem. Let $\{s_n\}$ and $\{t_n\}$ be sequences in $\mathbb{R}$ such that $s_n \to s$ and $t_n \to t$ where $s, t \in \mathbb{R}$. Then

(a) $s_n + t_n \to s + t$

(b) For all constants $c \in \mathbb{R}$, $cs_n \to c \cdot s$

(c) $s_n t_n \to s \cdot t$

(d) If $s \neq 0$ then $\frac{1}{s_n} \to \frac{1}{s}$

Proof of (a). We have that

$$\begin{aligned} d(s_n + t_n, s + t) &= |s_n + t_n - (s + t)| \\ &= |(s_n - s) + (t_n - t)| \\ &\leq |s_n - s| + |t_n - t| \end{aligned}$$

Given $\varepsilon > 0$, let $\varepsilon' = \frac{\varepsilon}{2} > 0$. We have $s_n \to s$ so there is $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$ we have $|s_n - s| < \varepsilon'$. We also have $t_n \to t$ so there is $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$ we have $|t_n - t| < \varepsilon'$. Let $n_0 = \max\{n_1, n_2\}$. Then $n \geq n_0 \Rightarrow n \geq n_1 \Rightarrow |s_n - s| < \varepsilon'$ and $n \geq n_0 \Rightarrow n \geq n_2 \Rightarrow |t_n - t| < \varepsilon'$. So for all $n \geq n_0$ we have

$$\begin{aligned} d(s_n + t_n, s + t) &\leq |s_n - s| + |t_n - t| \\ &< \varepsilon' + \varepsilon' = 2\varepsilon' = \varepsilon \end{aligned}$$

$\square$

Proof of (d). Let $\varepsilon_0 = \frac{|s|}{2}$ then $\varepsilon_0 > 0$. So there is $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$ we have $|s_n - s| < \varepsilon_0$. Let $n \geq n_1$, $|s| = |s - s_n + s_n| \leq |s - s_n| + |s_n| < \frac{|s|}{2} + |s_n|$. So for all $n \geq n_1$, $\frac{|s|}{2} < |s_n|$. In particular for all $n \geq n_1$, $s_n \neq 0$ so $\frac{1}{s_n}$ is defined. And also $\frac{1}{|s_n|} < \frac{2}{|s|}$. To show $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$, let $\varepsilon > 0$ be given. Let $\varepsilon' = \frac{\varepsilon |s|^2}{2} > 0$. We have $s_n \to s$ so there is $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$ we have $|s_n - s| < \varepsilon'$. Let $n_0 = \max\{n_1, n_2\}$ and $n \geq n_0$. Then

$$\begin{aligned} \left| \frac{1}{s_n} - \frac{1}{s} \right| &= \left| \frac{s - s_n}{s \cdot s_n} \right| = \frac{|s - s_n|}{|s||s_n|} \\ &< \frac{\varepsilon'}{|s|} \cdot \frac{1}{|s_n|} < \frac{\varepsilon'}{|s|} \cdot \frac{2}{|s|} \\ &= \frac{\varepsilon |s|^2}{2} \cdot \frac{2}{|s|^2} = \varepsilon \end{aligned}$$

$\square$

4.8 Theorem.

(a) Let $\{p_n\}$ be a sequence in $\mathbb{R}^k$ where $p_n = (x_1^n, \dots, x_k^n)$ and $p = (x_1, \dots, x_k) \in \mathbb{R}^k$. Then $p_n \to p \Leftrightarrow x_1^n \to x_1, x_2^n \to x_2, \dots, x_k^n \to x_k$.

(b) Let $\{p_n\}$, $\{q_n\}$ be two sequences in $\mathbb{R}^k$ and $\{\alpha_n\}$ be a sequence in $\mathbb{R}$. Assume $p_n \to p$, $q_n \to q$ in $\mathbb{R}^k$ and $\alpha_n \to \alpha$ in $\mathbb{R}$. Then $p_n + q_n \to p + q$ and $\alpha_n p_n \to \alpha p$.

Proof of (a). We need the following: If $q = (y_1, \dots, y_k) \in \mathbb{R}^k$ then for all $i = 1, 2, \dots, k$

$$
\begin{aligned}
&|y_i| \le \sqrt{y_1^2 + y_2^2 + \cdots + y_k^2} \le |y_1| + |y_2| + \cdots + |y_k| \\
&|y_i|^2 = y_i^2 \le y_1^2 + y_2^2 + \cdots + y_k^2 \\
&y_1^2 + y_2^2 + \cdots + y_k^2 \le (|y_1| + |y_2| + \cdots + |y_k|)^2
\end{aligned}
$$

Assume $p_n \to p$. Given $\varepsilon > 0$, we have $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, $d_2(p_n, p) < \varepsilon$. Let $n \ge n_0$. Then

$$
\begin{aligned}
&|x_1^n - x_1| \le \sqrt{(x_1^n - x_1)^2 + (x_2^n - x_2)^2 + \cdots + (x_k^n - x_k)^2} = d_2(p_n, p) < \varepsilon \\
&|x_2^n - x_2| \le \cdots < \varepsilon \\
&\qquad\qquad \vdots
\end{aligned}
$$

Conversely, assume $x_1^n \to x_1, x_2^n \to x_2, \dots, x_k^n \to x_k$. To show $p_n \to p$, let $\varepsilon > 0$ be given. Let $\varepsilon' = \frac{\varepsilon}{k} > 0$.

$$
\begin{aligned}
&x_1^n \to x_1 \text{ so we have } n_1 \text{ such that for all } n \ge n_1,\ |x_1^n - x_1| < \varepsilon' \\
&x_2^n \to x_2 \text{ so we have } n_2 \text{ such that for all } n \ge n_2,\ |x_2^n - x_2| < \varepsilon' \\
&\quad \vdots \\
&x_k^n \to x_k \text{ so we have } n_k \text{ such that for all } n \ge n_k,\ |x_k^n - x_k| < \varepsilon'
\end{aligned}
$$

Let $n_0 = \max\{n_1, n_2, \dots, n_k\}$. For all $n \ge n_0$

$$
\begin{aligned}
d_2(p_n, p) &= \sqrt{(x_1^n - x_1)^2 + (x_2^n - x_2)^2 + \cdots + (x_k^n - x_k)^2} \\
&\le |x_1^n - x_1| + |x_2^n - x_2| + \cdots + |x_k^n - x_k| \\
&< \varepsilon' + \varepsilon' + \cdots + \varepsilon' \\
&= k\varepsilon' = \varepsilon
\end{aligned}
$$

$\square$

4.2 Subsequences

4.9 Definition. Let $\{p_n\}$ be a sequence in X. Let $\{n_k\}$ be a sequence of natural numbers such that $n_1 < n_2 < n_3 < \cdots$ Then the sequence $\{p_{n_k}\}$ is called a *subsequence* of $\{p_n\}$.

4.10 Example. $\{p_1, p_3, p_7, p_{10}, p_{23}, \ldots\}$ is a subsequence of $\{p_n\}$. $n_1 = 2$, $n_2 = 3$, $n_3 = 7$, $n_4 = 10$, $n_5 = 23$ and so on.

4.11 Proposition. $p_n \to p \Leftrightarrow$ every subsequence of $\{p_n\}$ converges to p.

Proof.

($\Leftarrow$): Since $\{p_n\}$ is a subsequence of itself, $p_n \to p$.

($\Rightarrow$): Let $p_n \to p$. Let $\{p_{n_k}\}$ be an arbitrary subsequence of $\{p_n\}$. To show $p_{n_k} \to p$, let $\varepsilon > 0$ be given. Since $p_n \to p$, we have n_0 such that for all $n \geq n_0$, $d(p_n, p) < \varepsilon$. If $k \geq n_0$ then $n_k \geq k \geq n_0$ so $d(p_{n_k}, p) < \varepsilon$. $\square$

4.12 Remark. Limits of subsequences are limit points of the sequence.

4.13 Example. $X = \mathbb{R}^2$ and $p_n = \left(\frac{n+(-1)^n n+1}{n}, \frac{1}{n}\right)$

$$n \text{ is even } \Rightarrow p_n = \left(\frac{2n+1}{n}, \frac{1}{n}\right) \to (2,0)$$
$$n \text{ is odd } \Rightarrow p_n = \left(\frac{1}{n}, \frac{1}{n}\right) \to (0,0)$$

So the sequence $\{p_n\}$ has limit points $(2,0)$ and $(0,0)$.

4.14 Example. In $X = \mathbb{R}$, $x_n = n + (-1)^n n + \frac{1}{n}$

$$n \text{ is even } \Rightarrow x_n = 2n + \frac{1}{n} \to +\infty$$
$$n \text{ is odd } \Rightarrow x_n = \frac{1}{n} \to 0$$

We do not accept $+\infty$ as a limit since $+\infty$ is not a member of $\mathbb{R}$. So 0 is the only limit point of the sequence $\{x_n\}$ but $\{x_n\}$ is divergent since it is not bounded.

4.15 Theorem.

(a) Let (X, d) be a compact metric space and $\{p_n\}$ be any sequence in X. Then $\{p_n\}$ has a subsequence that converges to a point $p \in X$.

(b) Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.

Proof.

(a) Case 1: $\{p_n\}$ has only finitely many distinct terms. Then at least one term, say p_{n_0} is repeated infinitely many times, i.e. the sequence $\{p_n\}$ has a subsequence all of whose terms are p. Then the limit of this subsequence is $p = p_{n_0} \in X$.

Case 2: $\{p_n\}$ has infinitely many distinct terms. Then the set $E = \{p_1, p_2, p_3, \ldots\}$ is an infinite subset of the compact set X. So it has a limit point $p \in X$. Then p is the limit of a subsequence of $\{p_n\}$.

(b) Since $\{p_n\}$ is bounded, there is a k-cell I such that $\{p_n\} \subset I$. I is compact, so by (a), $\{p_n\}$ has a subsequence that converges to a point $p \in I$. $\square$

4.3 Cauchy Sequences

4.16 Definition. Let (X, d) be a metric space. A sequence $\{p_n\}$ in X is called a *Cauchy sequence* if for every $\varepsilon > 0$ we have $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, $d(p_n, p_m) < \varepsilon$.

4.17 Example. In $X = \mathbb{R}$

$$x_n = \int_1^n \frac{\cos t}{t^2} dt$$

Then $\{x_n\}$ is a Cauchy sequence in $\mathbb{R}$. Given n, m if $n = m$ then $x_n = x_m$

so $d(x_n, x_m) = |x_n - x_m| = 0 < \varepsilon$. If $n \neq m$, assume $m < n$.

$$\begin{aligned}
d(x_n - x_m) &= |x_n - x_m| \\
&= \left| \int_1^n \frac{\cos t}{t^2} dt - \int_1^m \frac{\cos t}{t^2} dt \right| \\
&= \left| \int_m^n \frac{\cos t}{t^2} dt \right| \\
&\leq \int_m^n \left| \frac{\cos t}{t^2} \right| dt \qquad \text{We know } \left| \frac{\cos t}{t^2} \right| = \frac{|\cos t|}{t^2} \leq \frac{1}{t^2} \\
&\leq \int_m^n \frac{1}{t^2} dt = -\frac{1}{t}\bigg|_m^n = -\frac{1}{n} + \frac{1}{m} \\
&\leq \frac{1}{m} \leq \frac{1}{n_0} < \varepsilon
\end{aligned}$$

Given $\varepsilon > 0$, choose $n_0 \in \mathbb{N}$ such that $\frac{1}{\varepsilon} < n_0$. Then for all $n, m \in \mathbb{N}$ with $n_0 \leq m \leq n$ we have $d(x_n, x_m) < \varepsilon$.

4.18 Example. $X = \mathbb{R}$, $x_n = \sqrt{n}$. If $n = m + 1$ then

$$\begin{aligned}
d(x_n, x_m) &= |x_{m+1} - x_m| \\
&= |\sqrt{m+1} - \sqrt{m}| = \sqrt{m+1} - \sqrt{m} \\
&= (\sqrt{m+1} - \sqrt{m}) \frac{\sqrt{m+1} + \sqrt{m}}{\sqrt{m+1} + \sqrt{m}} \\
&= \frac{1}{\sqrt{m+1} + \sqrt{m}} < \frac{1}{\sqrt{m}}
\end{aligned}$$

Given $\varepsilon > 0$, choose $n_0 \in \mathbb{N}$ such that $\frac{1}{\varepsilon^2} < n_0$. Then for all $m \geq n_0$ we have $d(x_{m+1}, x_m) < \varepsilon$.

So the distance between successive terms gets smaller as the index gets larger but this sequence $\{x_n\}$ is not a Cauchy sequence. For example, for $\varepsilon = 1$, consider

$$\begin{aligned}
d(x_m, x_{3m+1}) &= |\sqrt{m} - \sqrt{3m+1}| \\
&= \sqrt{3m+1} - \sqrt{m} = \sqrt{m + 2m + 1} - \sqrt{m} \\
&\geq \sqrt{(\sqrt{m}+1)^2} - \sqrt{m} \\
&\geq \sqrt{m} + 1 - \sqrt{m} \geq 1
\end{aligned}$$

4.19 Theorem. Let (X, d) be a metric space.

(a) Every convergent sequence in X is Cauchy.

(b) Every Cauchy sequence is bounded.

Proof.

(a) Assume $p_n \to p$. Show $\{p_n\}$ is Cauchy. Given $\varepsilon > 0$ let $\varepsilon' = \frac{\varepsilon}{2} > 0$. Since $p_n \to p$, there is $n_0 \in \mathbb{N}$ such that $d(p_n, p) < \varepsilon'$ for all $n \geq n_0$. Let $n, m \geq n_0$. Then

$$d(p_n, p_m) \leq d(p_n, p) + d(p, p_m) < \varepsilon' + \varepsilon' = 2\varepsilon' = \varepsilon$$

(b) Let $\{p_n\}$ be a Cauchy sequence in X. For $\varepsilon = 1$ there is $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $d(p_n, p_m) < 1$. Let $K = \max\{1, d(p_1, p_{n_0}), \ldots, d(p_{n_0 - 1}, p_{n_0})\}$ and $M = 2k$. Then we show that for all $n, m \in \mathbb{N}$, $d(p_n, p_m) \leq M$.

Case 1: $n, m \geq n_0$. Then

$$d(p_n, p_m) < 1 \leq K < M$$

Case 2: $n, m < n_0$. Then

$$d(p_n, p_m) \leq d(p_n, p_{n_0}) + d(p_{n_0}, p_m) \leq K + K = M$$

Case 3: $m < n_0$ and $n \geq n_0$. Then

$$d(p_n, p_m) \leq \underbrace{d(p_n, p_{n_0})}_{<1} + \underbrace{d(p_{n_0}, p_m)}_{\leq K} < 1 + K \leq K + K = M \qquad \square$$

Converse of (a) is not true in general.

4.20 Example. $X = (0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$ with $d(x, x') = |x - x'|$. $x_n = \frac{1}{2n}$. $\{x_n\}$ is a Cauchy sequence in X but $\{x_n\}$ has no limit in X.

4.21 Definition. A metric space (X, d) is said to be *complete* if every Cauchy sequence in (X, d) is convergent to some point $p \in X$.

4.22 Theorem.

(a) Every compact metric space (X, d) is complete.

(b) $(\mathbb{R}^k, d_2)$ is complete. ($(\mathbb{R}^k, d_1)$, $(\mathbb{R}^k, d_\infty)$ are also complete.)

Proof.

(a) Let (X,d) be a compact metric space and let $\{p_n\}$ be a Cauchy sequence in X. Then $\{p_n\}$ has a subsequence $\{p_{n_k}\}$ which converges to a point $p \in X$. Show $p_n \to p$. Let $\varepsilon > 0$ be given. Let $\varepsilon' = \frac{\varepsilon}{2} > 0$. $\{p_n\}$ is Cauchy, so there is $N_1 \in \mathbb{N}$ such that for all $n, m \geq N_1$ we have $d(p_n, p_m) < \varepsilon'$. $p_{n_k} \to p$, so there is N_2 such that for all $k \geq N_2$ we have $d(p_{n_k}, p) < \varepsilon'$. Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$

$$d(p_n, p) \leq \underbrace{d(p_n, p_{n_N})}_{<\varepsilon'} + \underbrace{d(p_{n_N}, p)}_{<\varepsilon'} < 2\varepsilon' = \varepsilon$$

(b) Let $\{p_n\}$ be a Cauchy sequence in $\mathbb{R}^k$. Then $\{p_n\}$ is bounded so there is a k-cell I such that $\{p_n\} \subset I$. (I, d_2) is compact. Then by (a), $\{p_n\}$ has a limit $p \in I \subset \mathbb{R}^k$. $\square$

4.23 Remark. $S \neq \emptyset$, $B(S)$ all bounded functions $f : S \to \mathbb{R}$.

$$d(f,g) = \sup\{|f(s) - g(s)| : s \in S\}$$

$(B(S), d)$ is complete.

4.24 Theorem. Let (X,d) be a complete metric space and $Y \neq \emptyset$ be a subset of X. The subspace (Y,d) is complete $\Leftrightarrow$ Y is a closed subset of X.

Proof.

($\Rightarrow$): Assume (Y,d) is complete and show Y is closed, i.e. $\overline{Y} \subset Y$. Let $p \in \overline{Y}$. Then there is a sequence $\{p_n\}$ in Y such that $p_n \to p$. Then $\{p_n\}$ is convergent in X. So $\{p_n\}$ is Cauchy in X. Since all $p_n \in Y$, $\{p_n\}$ is Cauchy in Y. Since Y is complete, there is an element $q \in Y$ such that $p_n \to q$. Then $p = q \in Y$. So $p \in Y$, i.e. $\overline{Y} \subset Y$.

($\Leftarrow$): Assume Y is closed. Show (Y,d) is complete. Let $\{p_n\}$ be a Cauchy sequence in Y. Then $\{p_n\}$ is a Cauchy sequence in X. Since (X,d) is complete, there is $p \in X$ such that $p_n \to p$. Then p is the limit of the sequence $\{p_n\}$ in Y. Then $p \in \overline{Y}$. Since Y is closed, $\overline{Y} = Y$. So $p \in Y$. So (Y,d) is complete. $\square$

4.25 Example. $X = \mathbb{R}^2$, $Y = \{(x,y) : x \geq 0,\ y \geq 0\}$. Then (Y, d_2) is complete.

Monotone Sequence Property In $\mathbb{R}$

Let $\{s_n\}$ be a sequence in $\mathbb{R}$. We say

$\{s_n\}$ is *increasing* if $s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_n \leq s_{n+1} \leq \cdots$

$\{s_n\}$ is *decreasing* if $s_1 \geq s_2 \geq s_3 \geq \cdots \geq s_n \geq s_{n+1} \geq \cdots$

$\{s_n\}$ is *monotone* if $\{s_n\}$ is either increasing or decreasing.

4.26 Theorem (Monotone Sequence Property)**.** Let $\{s_n\}$ be a monotone sequence in $\mathbb{R}$. Then $\{s_n\}$ is convergent $\Leftrightarrow$ $\{s_n\}$ is bounded.

Proof.

($\Rightarrow$): True for all sequences.

($\Leftarrow$): We do the proof for decreasing sequences. Let $s = \inf\{s_1, s_2, s_3, \ldots\}$. Show $\lim_{n\to\infty} s_n = s$. Let $\varepsilon > 0$ be given. Then $s + \varepsilon$ cannot be a lower bound for the set $\{s_1, s_2, s_3, \ldots\}$. Then there is s_{n_0} such that $s_{n_0} < s + \varepsilon$. Let $n \geq n_0$. $s - \varepsilon < s \leq s_n \leq s_{n_0} < s + \varepsilon$. For all $n \geq n_0$

$$\begin{aligned} s - \varepsilon < s_n &< s + \varepsilon \\ -\varepsilon < s_n - s &< \varepsilon \\ |s_n - s| &< \varepsilon \\ d(s_n, s) &< \varepsilon \end{aligned}$$

□

4.27 Example. Let $A > 0$ be fixed. Start with any $x_1 > 0$ and define

$$x_n = \frac{1}{2}\left(x_{n-1} + \frac{A}{x_{n-1}}\right) \qquad n = 2, 3, 4, \ldots$$

Then $\lim_{n\to\infty} x_n = \sqrt{A}$. We will show $x_2 \geq x_3 \geq x_4 \geq \cdots$ For $n \geq 2$

$$\begin{aligned} x_n^2 - A &= \frac{1}{4}\left(x_{n-1}^2 + \frac{A^2}{x_{n-1}^2} + 2A\right) - A \\ &= \frac{1}{4}\left(x_{n-1}^2 + \frac{A^2}{x_{n-1}^2} - 2A\right) \\ &= \frac{1}{4}\left(x_{n-1} - \frac{A}{x_{n-1}}\right)^2 \geq 0 \end{aligned}$$

So $x_n^2 \geq A$ for all $n \geq 2$. Since all $x_n > 0$, $x_n \geq \sqrt{A}$ for all $n \geq 2$. For $n \geq 2$

$$\begin{aligned} x_n - x_{n+1} &= x_n - \frac{1}{2}\left(x_n + \frac{A}{x_n}\right) \\ &= \frac{1}{2}\left(x_n - \frac{A}{x_n}\right) \\ &= \frac{1}{2}\frac{x_n^2 - A}{x_n} \geq 0 \end{aligned}$$

So $x_n \geq x_{n+1}$ for all $n \geq 2$. So $\{x_n\}_{n=2}^{\infty}$ is decreasing and bounded. So $\lim_{n\to\infty} x_n = x$ exists. Then we solve for x. We have that

$$\begin{array}{ccc} x_{n+1} & = & \frac{1}{2}\left(x_n + \frac{A}{x_n}\right) \\ \downarrow & & \downarrow \\ x & = & \frac{1}{2}\left(x + \frac{A}{x}\right) \end{array}$$

Then

$$\begin{aligned} 2x &= x + \frac{A}{x} \\ x^2 &= A \\ x &= \mp\sqrt{A} \end{aligned}$$

Since all $x_n > 0$, limit x cannot be negative. So $x = \sqrt{A}$.

4.4 Upper And Lower Limits

Let $\{x_n\}$ be a sequence in $\mathbb{R}$.

We write $\lim_{n\to\infty} x_n = +\infty$ (or $x_n \to +\infty$) if for every $M > 0$ we can find a natural number n_0 (depending on M in general) such that for all $n \geq n_0$ we have $M \leq x_n$.

We write $\lim_{n\to\infty} x_n = -\infty$ (or $x_n \to -\infty$) if for every $M < 0$ we can find a natural number n_0 (depending on M in general) such that for all $n \geq n_0$ we have $x_n \leq M$.

In either case, we say $\{x_n\}$ is divergent.

4.28 Example. $x_n = n + \frac{1}{n}$ and $\lim_{n\to\infty} x_n = +\infty$. $\{x_n\}$ is divergent.

4.29 Definition. Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Let E be the set of all subsequential limits of $\{x_n\}$. Then $E \subset \mathbb{R} \cup \{-\infty, +\infty\}$.

4.30 Example. $x_n = n + (-1)^n n + \frac{1}{n}$. Subsequential limits are $+\infty$ and 0. So $E = \{0, +\infty\}$.

4.31 Definition. Let $x^* = \sup E$ and $x_* = \inf E$ (Considered in the set of extended real numbers.)

x^* is called the *upper limit* (or *limit superior*) of $\{x_n\}$ and it is denoted by

$$x^* = \limsup_{n\to\infty} x_n = \overline{\lim_{n\to\infty}}\, x_n$$

x_* is called the *lower limit* (or *limit inferior*) of $\{x_n\}$ and it is denoted by

$$x_* = \liminf_{n\to\infty} x_n = \underline{\lim_{n\to\infty}}\, x_n$$

4.32 Example. The function $\Pi : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ defined by $\Pi(r,s) = 2^{r-1}(2s-1)$, is 1-1 and onto. Let s be fixed. $\mathbb{N}_s = \{2^{r-1}(2s-1) : r = 1,2,3,\ldots\} =$

$r\backslash s$	1	2	3	4	$\cdots$
1	1	3	5	7	$\cdots$
2	2	6	10	14	$\cdots$
3	4	12	20	28	$\cdots$
4	8	24	40	56	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$\{2s-1, 2(2s-1), 4(2s-1), \ldots\}$ Then for $s \neq s'$, $\mathbb{N}_s \cap \mathbb{N}_{s'} = \emptyset$. Also $\bigcup_{s=1}^{\infty} \mathbb{N}_s = \mathbb{N}$. Define a sequence $\{x_n\}$ in $\mathbb{R}$ as follows: Given $n \in \mathbb{N}$, there is a unique s such that $n \in \mathbb{N}_s$. Define $x_n = \frac{sn}{n+1}$. What are the subsequential limits of $\{x_n\}$? What are $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$?

$$\text{If } n \in \mathbb{N}_1 = \{1,2,4,8,\ldots\} \text{ then } x_n = \frac{n}{n+1} \to 1$$
$$\text{If } n \in \mathbb{N}_2 = \{3,6,12,24,\ldots\} \text{ then } x_n = \frac{2n}{n+1} \to 2$$
$$\vdots$$
$$\text{If } n \in \mathbb{N}_s = \{\cdots\} \text{ then } x_n = \frac{sn}{n+1} \to s$$

So all $1,2,3,\ldots$ are subsequential limits. Then we have $\{1,2,3,\ldots\} \subset E$. Then $\sup E = +\infty$, i.e. $\limsup_{n\to\infty} x_n = +\infty$.

Since $x_n = \frac{s_n}{n+1}$, $s \geq 1 \Rightarrow x_n \geq \frac{n}{n+1}$ so all $x_n \geq \frac{1}{2}$. So if x is a subsequential limit of $\{x_n\}$ then $x \geq \frac{1}{2}$. Can we have a subsequential limit x such that $\frac{1}{2} \leq x \leq 1$? If $n \in \mathbb{N}_s$ where $s \geq 2$ then $x_n = \frac{s_n}{n+1} \geq \frac{2n}{n+1} \geq 1$. If $n \in \mathbb{N}_1$ then $x_n = \frac{n}{n+1} \to 1$. So 1 is the smallest subsequential limit of $\{x_n\}$. Thus $\liminf_{n\to\infty} x_n = 1$.

Properties

(i) $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$

(ii) $\underbrace{\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n}_{\text{call this } x} \Leftrightarrow \lim_{n\to\infty} x_n = x$ (Here $x \in \mathbb{R}$ or $x = +\infty$ or $x = -\infty$)

(iii) Let $x \in \mathbb{R}$, i.e. $x \neq \mp\infty$. We have that $\limsup_{n\to\infty} x_n = x \Leftrightarrow$

  (a) For every $\varepsilon > 0$ there is a natural number n_0 such that for all $n \geq n_0$, $x_n < x + \varepsilon$

  and

  (b) For every $\varepsilon > 0$ there are infinitely many n such that $x - \varepsilon < x_n$

4.33 Theorem (Squeeze Property or Sandwich Property)**.** Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ be three sequences in $\mathbb{R}$ such that $x_n \leq y_n \leq z_n$ for all n. Assume that $\{x_n\}$ and $\{z_n\}$ are convergent and $\underbrace{\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n}_{\text{call this } c}$. Then $\{y_n\}$ is convergent and $\lim_{n\to\infty} y_n = c$.

Proof. Given $\varepsilon > 0$

$x_n \to c$, so there is $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$, $|x_n - c| < \varepsilon$

$z_n \to c$, so there is $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$, $|z_n - c| < \varepsilon$

Let $n_0 = \max\{n_1, n_2\}$ and $n \geq n_0$. Show $|y_n - c| < \varepsilon$

If $c \leq y_n$ then $|y_n - c| = y_n - c \leq z_n - c \leq |z_n - c| < \varepsilon$

If $c > y_n$ then $|y_n - c| = c - y_n \leq c - x_n \leq |c - x_n| = |x_n - c| < \varepsilon$ □

Some Special Sequences In $\mathbb{R}$

4.34 Theorem.

(a) If $p > 0$ constant then $\lim_{n\to\infty} \frac{1}{n^p} = 0$

(b) If $p > 0$ constant then $\lim_{n\to\infty} \sqrt[n]{p} = 1$

(c) $\lim_{n\to\infty} \sqrt[n]{n} = 1$

(d) If $p > 0$ and $\alpha \in \mathbb{R}$ are constants then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$

Note: This is usually expressed as polynomials tend to increase slower than exponentials.

(e) If x is constant and $|x| < 1$, i.e. $-1 < x < 1$ then $\lim_{n\to\infty} x^n = 0$

Proof of (b).
If $p = 1$ then $\sqrt[n]{p} = 1$ for all n, so $\lim_{n\to\infty} \sqrt[n]{p} = 1$.

If $p > 1$ let $x_n = \sqrt[n]{p} - 1$. Then $x_n > 0$ for all n.

$$p = (1 + x_n)^n = 1 + nx_n + \underbrace{\frac{n(n-1)}{2}x_n^2 + \cdots + x_n^n}_{\text{positive}}$$

So $p \geq 1 + nx_n$. So $0 < x_n < \frac{p-1}{n}$. By sandwich property, $\lim_{n\to\infty} x_n = 0$. Then $\sqrt[n]{p} = 1 + x_n \to 1$.

If $p < 1$ then $\sqrt[n]{\frac{1}{p}} \to 1$ by the previous case so $\sqrt[n]{p} \to 1$. $\square$

Proof of (c). Let $x_n = \sqrt[n]{n} - 1$. Then $x_n \geq 0$ for all n.

$$n = (1 + x_n)^n = 1 + nx_n + \frac{n(n-1)}{2}x_n^2 + \underbrace{\frac{n(n-1)(n-2)}{6}x_n^3 + \cdots + x_n^n}_{\geq 0}$$

$$\geq \frac{n(n-1)}{2}x_n^2$$

So $n \geq \frac{n(n-1)}{2}x_n^2 \Rightarrow 0 \leq x_n^2 \leq \frac{2}{n-1} \Rightarrow 0 \leq x_n \leq \sqrt{\frac{2}{n-1}}$. By sandwich theorem, $\lim_{n\to\infty} x_n = 0$. Then $\sqrt[n]{n} = 1 + x_n \to 1$. $\square$

Proof of (d). If $\alpha \leq 0$ we have $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$. So assume $\alpha > 0$. Fix a natural number k such that $\alpha < k$. Then for $n \geq 2k$

$$\underbrace{n(n-1)\cdots(n-k+1)}_{k \text{ terms}} > \underbrace{\frac{n}{2}\cdot\frac{n}{2}\cdots\frac{n}{2}}_{k \text{ terms}} = \left(\frac{n}{2}\right)^k$$

$$
\begin{aligned}
(1+p)^n &= \sum_{\ell=0}^{n} \binom{n}{\ell} p^\ell \cdot 1^{n-\ell} \\
&> \binom{n}{k} p^k = \frac{n(n-1)\cdots(n-k+1)}{k!} p^k \\
&> \frac{\left(\frac{n}{2}\right)^k}{k!} p^k = \frac{n^k}{2^k k!} p^k = \frac{n^{k-\alpha} n^\alpha}{2^k k!} p^k \\
\frac{2^k k!}{p^k} \frac{1}{n^{k-\alpha}} &> \frac{n^\alpha}{(1+p)^n} > 0
\end{aligned}
$$

By sandwich property, we have $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$. □

4.5 Series

4.35 Definition. Given a sequence $\{a_n\}$ in $\mathbb{R}$, the symbol $\sum_{n=1}^{\infty} a_n$ is called an (infinite) *series*. Given a series $\sum_{n=1}^{\infty} a_n$ we define the following sequence $\{s_n\}$

$$
\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
s_3 &= a_1 + a_2 + a_3 \\
&\vdots \\
s_n &= a_1 + a_2 + \cdots + a_n
\end{aligned}
$$

$\{s_n\}$ is called the sequence of partial sums.

If $\lim_{n\to\infty} s_n = s$ exists in $\mathbb{R}$ ($s = \mp\infty$ is not acceptable) we say the series is *convergent* and has sum $= s$. We write $\sum_{n=1}^{\infty} a_n = s$.

If $\lim_{n\to\infty} s_n = \mp\infty$ or $\lim_{n\to\infty} s_n$ does not exist we say that the series $\sum_{n=1}^{\infty} a_n$ is *divergent*.

4.36 Example. $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2-1} = \frac{1}{2^2-1} + \frac{1}{3^2-1} + \frac{1}{4^2-1} + \cdots$ Then

$$
a_n = \frac{1}{(n+1)^2 - 1} = \frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}
$$

Then $A = \frac{1}{2}$ and $B = -\frac{1}{2}$. We get

$$
\frac{\frac{1}{2}}{n} - \frac{\frac{1}{2}}{n+2} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right) = \frac{1}{2}\frac{n+2-n}{n(n+2)}
$$

So $a_n = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)$. Then

$$\begin{aligned} s_n &= a_1 + a_2 + \cdots + a_n \\ &= \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right) \end{aligned}$$

Cancellation pattern: (|X) + (|) + (X|)+ $\cdots$ Then

$$s_n = \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{1}{2}\left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)$$

Then $\lim_{n\to\infty} s_n = \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}$ so $s = \frac{3}{4}$ i.e. $\frac{1}{2^2-1} + \frac{1}{3^2-1} + \cdots = \frac{3}{4}$

4.37 Example. Let $r \in \mathbb{R}$ be a constant. Consider $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots$ geometrical series.

$$\begin{aligned} s_n &= 1 + r + r^2 + \cdots + r^n \\ rs_n &= \underbrace{r + r^2 + \cdots + r^n}_{s_n - 1} + r^{n+1} \\ rs_n &= s_n - 1 + r^{n+1} \\ 1 - r^{n+1} &= s_n - rs_n \\ s_n &= \frac{1 - r^{n+1}}{1 - r} \quad \text{if } r \neq 1 \end{aligned}$$

If $|r| < 1$, i.e. $-1 < r < 1$ then $r^{n+1} \to 0$ so $\lim_{n\to\infty} s_n = \frac{1}{1-r}$

If $r = 1$ then $s_n = n + 1 \to +\infty$

For any other value of r, $\lim_{n\to\infty} s_n$ does not exist. So the geometrical series is convergent only for $-1 < r < 1$.

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r} \quad \text{if } -1 < r < 1$$

4.38 Theorem (Cauchy Criterion)**.** $\sum_{n=1}^{\infty} a_n$ is convergent $\Leftrightarrow$ for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ with $n \geq m$ we have $|\sum_{k=m}^{n} a_k| = |a_m + a_{m+1} + \cdots + a_n| < \varepsilon$.

Proof. $\sum_{n=1}^{\infty} a_n$ is convergent then $\{s_n\}$ is convergent. So $\{s_n\}$ is Cauchy. That is, for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ with $n \geq m$ we have $|s_n - s_{m-1}| < \varepsilon$.

$$s_n - s_{m-1} = a_1 + a_2 + \cdots + a_n - (a_1 + \cdots + a_{m-1}) = a_m + \cdots + a_n \quad \square$$

4.39 Theorem. If $\sum a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$.

Proof. $a_n = s_n - s_{n-1} \to s - s = 0$. $\square$

4.40 Example. $\sum_{n=1}^{\infty}(-1)^n = (-1)+1+(-1)+1+(-1)+\cdots$ is divergent since $\lim_{n\to\infty}(-1)^n$ does not exist. So $\lim_{n\to\infty}(-1)^n \neq 0$.

$$s_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$$

4.41 Definition. A series $\sum_{n=1}^{\infty} a_n$ is said to be *non-negative* if there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $a_n \geq 0$.

4.42 Theorem. Let $\sum a_n$ be a non-negative series. Then $\sum a_n$ is convergent $\Leftrightarrow$ $\{s_n\}$ is bounded.

Proof. There is n_0 such that for all $n \geq n_0$ we have $a_n \geq 0$.

$$\begin{aligned} s_{n_0} &= s_{n_0-1} + a_{n_0} \geq s_{n_0-1} \\ s_{n_0+1} &= s_{n_0} + a_{n_0+1} \geq s_{n_0} \\ &\vdots \\ s_{n+1} &= s_n + a_{n+1} \geq s_n \\ &\vdots \\ s_{n_0} &\leq s_{n_0+1} \leq \cdots \leq s_n \leq s_{n+1} \leq \cdots \end{aligned}$$

By monotone sequence property (since $\{s_n\}_{n=n_0}^{\infty}$ is increasing), $\{s_n\}$ is convergent $\Leftrightarrow$ $\{s_n\}$ is bounded. $\square$

4.43 Theorem (Comparison Test)**.**

(a) Suppose there is n_0 such that for all $n \geq n_0$ $|a_n| \leq c_n$ and $\sum c_n$ is convergent. Then $\sum a_n$ is also convergent.

(b) Suppose there is n_0 such that for all $n \geq n_0$ $a_n \geq d_n \geq 0$ and $\sum d_n$ is divergent. Then $\sum a_n$ is also divergent.

Proof.

(a) Use Cauchy criterion. Let $\varepsilon > 0$ be given. Since $\sum c_n$ is convergent, there is $n_1 \in \mathbb{N}$ such that for all $n, m \geq n_1$ with $m \leq n$ we have $|\sum_{k=m}^{n} c_k| < \varepsilon$. Let $n_2 = \max\{n_0, n_1\}$. Let $n \geq m \geq n_2$. Then

$$\left|\sum_{k=m}^{n} a_k\right| = |a_m + a_{m+1} + \cdots + a_n| \leq \underbrace{|a_m|}_{\leq c_m} + \underbrace{|a_{m+1}|}_{\leq c_{m+1}} + \cdots + \underbrace{|a_n|}_{\leq c_n}$$
$$\leq c_m + c_{m+1} + \cdots + c_n = |c_m + c_{m+1} + \cdots + c_n| < \varepsilon$$

(b) This follows from (a). If $\sum a_n$ were convergent, then by (a), $\sum d_n$ would be convergent. $\square$

4.44 Theorem (Cauchy Condensation Test)**.** Suppose $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$. Then $\sum_{n=1}^{\infty} a_n$ is convergent $\Leftrightarrow \sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots$ is convergent.

Proof. Let $s_n = a_1 + a_2 + \cdots + a_n$ and $t_n = a_1 + 2a_2 + \cdots + 2^n a_{2^n}$.

$$\begin{aligned}
t_n &= a_1 + 2a_2 + 4a_4 + \cdots + 2^n a_{2^n} \\
&= a_1 + (a_2 + a_2) + (a_4 + a_4 + a_4 + a_4) + \cdots + \underbrace{(a_{2^n} + a_{2^n} + \cdots + a_{2^n})}_{2^n \text{ terms}} \\
&\geq a_1 + a_2 + a_3 + \cdots + a_{2^{n+1}-1} = s_{2^{n+1}-1} \\
s_{2^n} &= a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \cdots + \underbrace{(a_{2^{n-1}+1} + \cdots + a_{2^n})}_{2^{n-1} \text{ terms}} \\
&\geq \frac{1}{2}a_1 + a_2 + 2a_4 + 4a_8 + \cdots + 2^{n-1} a_{2^n} \\
&= \frac{1}{2}(a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots + 2^n a_{2^n}) = \frac{1}{2} t_n
\end{aligned}$$

Then $s_{2^{n+1}-1} \leq t_n \leq 2 s_{2^n}$. So $\sum a_n$ is convergent $\Leftrightarrow \{s_n\}$ is bounded $\Leftrightarrow \{t_n\}$ is bounded $\Leftrightarrow \sum 2^k a_{2^k}$ is convergent. $\square$

4.45 Example. Let $p > 0$ be constant and consider the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$. Then $a_n = \frac{1}{n^p} > 0$ and $a_1 \geq a_2 \geq a_3 \geq \cdots$

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = \sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^p} = \sum_{k=0}^{\infty} 2^{k-kp} = \sum_{k=0}^{\infty} (2^{1-p})^k$$

Geometric series with $r = 2^{1-p} > 0$. It is convergent $\Leftrightarrow r < 1$, i.e. $2^{1-p} < 1 \Leftrightarrow p > 1$. If $p \leq 0$ let $q = -p \geq 0$. $a_n = \frac{1}{n^p} = \frac{1}{n^{-q}} = n^q$

then $\lim_{n\to\infty} a_n \neq 0$. So $\sum \frac{1}{n^p}$ is divergent.

Summary: Let p be a constant. Then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent $\Leftrightarrow p > 1$.

For example, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ are convergent but $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ are divergent. $\sum_{n=1}^{\infty} \frac{1}{n^{1.00001}}$ is convergent. $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ is not a p-series since the exponent $1 + \frac{1}{n}$ is not a constant.

4.46 Remark.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

The Number e

The series

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

is convergent. For $n \geq 2$

$$a_n = \frac{1}{n!} = \frac{1}{\underbrace{2 \cdot 3 \cdots n}_{n-1 \text{ factors}}} \leq \frac{1}{2^{n-1}}$$

$\sum \frac{1}{2^{n-1}} = 2 \sum \left(\frac{1}{2}\right)^n$ is convergent. So by comparison test, $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{1}{n!}$ is convergent.

4.47 Definition.

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

4.48 Theorem. e is not rational.

Proof. Suppose e is rational. Then $e = \frac{p}{q}$ where p, q are natural numbers.

Since all $a_n > 0$, we have $s_n < e$ for all n. We have $0 < e - s_q < \frac{1}{q!q}$

$$\begin{aligned} e - s_q &= \frac{1}{(q+1)!} + \frac{1}{(q+2)!} + \frac{1}{(q+3)!} + \cdots \\ &= \frac{1}{(q+1)!}\left[1 + \frac{1}{q+2} + \frac{1}{(q+2)(q+3)} + \cdots\right] \\ &< \frac{1}{(q+1)!}\left[1 + \frac{1}{q+1} + \frac{1}{(q+1)(q+1)} + \cdots\right] \\ &= \frac{1}{(q+1)!}\sum_{n=0}^{\infty}\left(\frac{1}{q+1}\right)^n \quad \text{Geometrical series with } r = \frac{1}{q+1} \\ &< \frac{1}{(q+1)!}\frac{1}{1-\frac{1}{q+1}} = \frac{1}{(q+1)!}\frac{q+1}{q} = \frac{1}{q!q} \end{aligned}$$

Then $0 < q!(e - s_q) < \frac{1}{q}$. $p = e \cdot q$ is an integer. So $q! \cdot e = 1 \cdot 2 \cdots (q-1) \cdot q \cdot e$ is an integer. $q!s_q = q!\left(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{q!}\right)$ is also an integer. So $q!e - qs_q$ is an integer. Also, $\frac{1}{q} \leq 1$ so $0 < q!e - q!s_q < 1$. Contradiction. $\square$

4.49 Remark. e is not even an algebraic number.

A real number r is called an *algebraic number* if there is a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with integer coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$ such that $P(r) = 0$.

4.50 Example. $r = \sqrt{2}$ is algebraic. $P(x) = x^2 - 2$ then $P(r) = 0$.

A real number that is not algebraic is called *transcendental*. e, π are transcendental numbers.

4.51 Theorem (Root Test)**.** Given $\sum a_n$ let $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Then $0 \leq \alpha \leq +\infty$.

(a) $\alpha < 1 \Rightarrow \sum a_n$ is convergent

(b) $\alpha > 1 \Rightarrow \sum a_n$ is divergent

(c) $\alpha = 1 \Rightarrow$ No information

Proof.

(a) Find β such that $\alpha < \beta < 1$. Then there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $\sqrt[n]{|a_n|} < \beta$. That is, $|a_n| < \beta^n$ for all $n \geq n_0$. $\sum \beta^n$ is convergent (geometrical series, $0 < \beta < 1$) so by comparison test, $\sum a_n$ is convergent.

(b) We have $\alpha > 1$. Since $\limsup$ is the largest subsequential limit, we can find a subsequence $\{n_k\}$ of natural numbers such that $\sqrt[n_k]{|a_{n_k}|} \to \alpha$. Since $\alpha > 1$, we have $\sqrt[n_k]{|a_{n_k}|} > 1 \Rightarrow |a_{n_k}| > 1$. Then "$\lim_{n\to\infty} a_n = 0$" cannot be true. So $\sum a_n$ is divergent.

(c) $\sum \frac{1}{n}$ is divergent and $\alpha = 1$. (in fact $\lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{n}\right|} = 1$)

$\sum \frac{1}{n^2}$ is convergent and $\alpha = 1$. (in fact $\lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{n^2}\right|} = 1$) □

4.52 Example. Consider $\sum_{n=1}^{\infty} \frac{(-2)^n}{n}$ Then $a_n = \frac{(-2)^n}{n}$. We apply root test. $\sqrt[n]{|a_n|} = \sqrt[n]{\frac{2^n}{n}} = \frac{2}{\sqrt[n]{n}}$. We have $\alpha = \lim_{n\to\infty} \frac{2}{\sqrt[n]{n}} = 2$. $\alpha > 1$ so $\sum \frac{(-2)^n}{n}$ is divergent.

4.53 Theorem (Ratio Test)**.** Let $\sum a_n$ be an arbitrary series.

(a) If $\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$ then $\sum a_n$ is convergent.

(b) If $\liminf_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1$ then $\sum a_n$ is divergent.

(c) If $\liminf_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \leq 1 \leq \limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ then no information.

Proof. Omitted. Similar to the proof of the root test.

4.54 Example. Consider $\sum a_n$ where

$$a_n = \begin{cases} \frac{n^2}{10^n} & \text{if } n \text{ is odd} \\ \frac{n^3}{100^n} & \text{if } n \text{ is even} \end{cases}$$

$a_n > 0$ for all n so $|a_n| = a_n$. Then

$$\frac{|a_{n+1}|}{|a_n|} = \frac{a_{n+1}}{a_n} = \begin{cases} \frac{\frac{(n+1)^3}{100^{n+1}}}{\frac{n^2}{10^n}} & \text{if } n \text{ is odd} \\ \frac{\frac{(n+1)^2}{10^{n+1}}}{\frac{n^3}{100^n}} & \text{if } n \text{ is even} \end{cases}$$

Then

$$\frac{|a_{n+1}|}{|a_n|} = \begin{cases} \frac{(n+1)^3}{n^2}\frac{1}{10^{n+2}} & \text{if } n \text{ is odd} \\ \frac{(n+1)^2}{n^3}10^{n-1} & \text{if } n \text{ is even} \end{cases}$$

So $\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = +\infty$ and $\liminf_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 0$. So ratio test gives no information. Then try root test.

$$\sqrt[n]{|a_n|} = \begin{cases} \frac{(\sqrt[n]{n})^2}{10} & \text{if } n \text{ is odd} \\ \frac{(\sqrt[n]{n})^3}{100} & \text{if } n \text{ is even} \end{cases}$$

$\frac{1}{10}$ and $\frac{1}{100}$ are the only subsequential limits of $\{\sqrt[n]{|a_n|}\}$. We have $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{10} < 1$. So by root test, the series $\sum a_n$ is convergent.

4.55 Remark. Root test has wider scope.

Ratio test shows convergence $\Rightarrow$ Root test shows convergence

Root test gives no information $\Rightarrow$ Ratio test gives no information

4.56 Theorem. Let $\{c_n\}$ be any sequence of positive numbers. Then

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \le \liminf_{n\to\infty} \sqrt[n]{c_n} \le \limsup_{n\to\infty} \sqrt[n]{c_n} \le \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}$$

Raabe's Test: Let $\sum a_n$ be a series of real numbers.

(a) $\liminf_{n\to\infty} n\left(\frac{|a_n|}{|a_{n+1}|} - 1\right) = p$ If $1 < p$ then $\sum a_n$ is convergent.

(b) $\limsup_{n\to\infty} n\left(\frac{|a_n|}{|a_{n+1}|} - 1\right) = q$ If $q < 1$ then $\sum a_n$ is divergent.

4.57 Example. Consider $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}$

$$\frac{|a_{n+1}|}{|a_n|} = \frac{a_{n+1}}{a_n} = \frac{\frac{1\cdot 3\cdots(2n-1)\cdot(2n+1)}{2\cdot 4\cdots(2n)\cdot(2n+2)}}{\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}} = \frac{2n+1}{2n+2} \to 1$$

We have $\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \liminf_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$. So ratio test gives no information. Then try Raabe's test.

$$n\left(\frac{|a_n|}{|a_{n+1}|} - 1\right) = n\left(\frac{2n+2}{2n+1} - 1\right) = n\frac{1}{2n+1} \to \frac{1}{2}$$

We have $\limsup_{n\to\infty} n\left(\frac{|a_n|}{|a_{n+1}|} - 1\right) = \frac{1}{2} < 1$. So the series is divergent.

4.58 Definition (Power Series)**.** Let $\{c_n\}$ be a fixed sequence of real numbers and $x \in \mathbb{R}$ be a variable. The series

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots = \sum_{n=0}^{\infty} c_n x^n$$

is called a *power series*.

4.59 Example. Let $c_n = \frac{1}{n!}$. Consider $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

General Question: Given a power series, find the set of all x for which the power series is convergent.

We apply root test.

$$\limsup_{n\to\infty} \sqrt[n]{|c_n x^n|} = \limsup_{n\to\infty} |x| \sqrt[n]{|c_n|} = |x| \underbrace{\limsup_{n\to\infty} \sqrt[n]{|c_n|}}_{\text{call } \alpha} = |x|\alpha$$

If $|x|\alpha < 1$, i.e. $|x| < \frac{1}{\alpha}$ then $\sum c_n x^n$ is convergent.

If $|x|\alpha > 1$, i.e. $|x| > \frac{1}{\alpha}$ then $\sum c_n x^n$ is divergent.

4.60 Theorem. With any power series $\sum c_n x^n$ is associated a *radius of convergence* R, $0 \le R \le +\infty$ such that

(i) The series converges for all x with $|x| < R$

(ii) The series diverges for all x with $|x| > R$

$R = \frac{1}{\alpha}$ where $\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}$ or $\alpha = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}$ if this limit exists. If $\alpha = 0$ then $R = +\infty$. If $\alpha = +\infty$ then $R = 0$. (For $R = 0$, it means the series $\sum c_n x^n$ converges only for $x = 0$.)

4.61 Example. Consider $\sum_{n=0}^{\infty} n! x^n = 1 + x + 2! x^2 + 3! x^3 + \cdots$ Then $c_n = n!$

$$\alpha = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} = \lim_{n\to\infty} (n+1) = +\infty$$

So $R = 0$ and $\sum n! x^n$ converges only for $x = 0$.

4.62 Example. Consider $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ Then $c_n = \frac{1}{n!}$

$$\alpha = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

So $R = +\infty$ and $\sum \frac{x^n}{n!}$ converges for all $x \in \mathbb{R}$ with $|x| < +\infty$, i.e. for all $x \in \mathbb{R}$.

4.63 Example. Consider $\sum_{n=1}^{\infty} \frac{x^n}{n}$. Then $c_n = \frac{1}{n}$

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{\left|\frac{1}{n}\right|} = \limsup_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1$$

So $R = 1$ and

The series $\sum \frac{x^n}{n}$ is convergent for all x with $|x| < 1$, i.e. $-1 < x < 1$

The series $\sum \frac{x^n}{n}$ is divergent for all x with $|x| > 1$, i.e. $x < -1$, $1 < x$

No information for $|x| = 1$, i.e. $x = \mp 1$

If $x = 1$, $\sum_{n=1}^{\infty} \frac{1^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. (p-series with $p = 1$)

If $x = -1$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots$

Abel's Partial Summation Formula: Let $\{a_n\}$ and $\{b_n\}$ be two sequences.

$$A_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

Then

$$\sum_{k=1}^{n+1} a_k b_k = A_{n+1} b_{n+1} - \sum_{k=1}^{n} A_k (b_{k+1} - b_k)$$

Proof. Let $A_0 = 0$. Then $a_k = A_k - A_{k-1}$, $k = 1, 2, \ldots$

$$\begin{aligned}
\sum_{k=1}^{n+1} a_k b_k &= \sum_{k=1}^{n+1} (A_k - A_{k-1})\, b_k \\
&= \sum_{k=1}^{n+1} A_k b_k - \underbrace{\sum_{k=1}^{n+1} A_{k-1} b_k}_{\sum_{k=1}^{n} A_k b_{k+1}} \\
&= \sum_{k=1}^{n} A_k b_k + A_{n+1} b_{n+1} - \sum_{k=1}^{n} A_k b_{k+1} \\
&= A_{n+1} b_{n+1} + \sum_{k=1}^{n} A_k (b_k - b_{k+1})
\end{aligned}$$ □

4.64 Theorem (Dirichlet's Test)**.** Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers. $A_n = \sum_{k=1}^{n} a_k = a_1 + \cdots + a_n$. Assume

(a) The sequence $\{A_n\}$ is bounded

(b) $b_1 \geq b_2 \geq b_3 \geq \cdots$

(c) $\lim_{n\to\infty} b_n = 0$

Then $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

Proof. Let $s_n = a_1 b_1 + \cdots + a_n b_n$. Show $\{s_n\}$ converges in $\mathbb{R}$. By Abel's formula

$$s_{n+1} = A_{n+1}b_{n+1} - \sum_{k=1}^{n} A_k(b_{k+1} - b_k)$$

$\{A_n\}$ is bounded, so there is a constant $M > 0$ such that $|A_n| \leq M$ for all n.

$$-M \leq A_{n+1} \leq M$$

If we multiply by $b_{n+1} \geq 0$ we get

$$-Mb_{n+1} \leq A_{n+1}b_{n+1} \leq Mb_{n+1}$$

So $\lim_{n\to\infty} A_{n+1}b_{n+1} = 0$. Next, show $\lim_{n\to\infty} \left(\sum_{k=1}^{n} A_k(b_{k+1} - b_k)\right)$ exists in $\mathbb{R}$. This is the n-th partial sum of the series $\sum_{n=1}^{\infty} A_n(b_{n+1} - b_n)$. So show the series $\sum_{n=1}^{\infty} A_n(b_{n+1} - b_n)$ is convergent. Use Cauchy criterion. Let $\varepsilon > 0$ be given. Let $\varepsilon' = \frac{\varepsilon}{2M} > 0$. Since $\lim_{n\to\infty} b_n = 0$, we have $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $|b_n| = |b_n - 0| < \varepsilon'$. Let $n, m \geq n_0$ and $n \geq m$.

$$\begin{aligned}
\left|\sum_{k=m}^{n} A_k(b_{k+1} - b_k)\right| &\leq \sum_{k=m}^{n} \underbrace{|A_k|}_{\leq M} |b_{k+1} - b_k| \leq M \sum^{n} k = m \underbrace{|b_{k+1} - b_k|}_{b_k - b_{k+1}} \\
&= M\left((b_m - b_{m+1}) + (b_{m+1} - b_{m+2}) + \cdots + (b_n - b_{n+1})\right) \\
&= M(b_m - b_{n+1}) \leq M|b_m - b_{n+1}| \leq M(\underbrace{|b_m|}_{<\varepsilon'} - \underbrace{|b_{n+1}|}_{<\varepsilon'}) \\
&< M 2\varepsilon' = \varepsilon
\end{aligned}$$

□

4.65 Example. Consider

$$\frac{1+\frac{1}{1^2}}{\sqrt{1}} + \frac{2}{\sqrt{2}} - \frac{3}{\sqrt{3}} + \frac{1+\frac{1}{4^2}}{\sqrt{4}} + \frac{2}{\sqrt{5}} - \frac{3}{\sqrt{6}} + \frac{1+\frac{1}{7^2}}{\sqrt{7}} + \frac{2}{\sqrt{8}} - \frac{3}{\sqrt{9}} + \cdots$$

Then $b_1 = \frac{1}{\sqrt{1}}$, $b_2 = \frac{1}{\sqrt{2}}$, $b_3 = \frac{1}{\sqrt{3}} \cdots b_n = \frac{1}{\sqrt{n}} \cdots$ $\{b_n\}$ satisfies (b) and (c). Also $a_1 = 1 + \frac{1}{1^2}$, $a_2 = 2$, $a_3 = -3$, $a_4 = 1 + \frac{1}{4^2}$, $a_5 = 2$, $a_6 = 3$, $a_7 = 1 + \frac{1}{7^2}$, $a_8 = 2$, $a_9 = -3$, $\ldots$

Then $A_1 = 1 + \frac{1}{1^2}$, $A_2 = 3 + \frac{1}{1^2}$, $A_3 = \frac{1}{1^2}$, $A_4 = \frac{1}{1^2} + 1 + \frac{1}{4^2}$, $A_5 = \frac{1}{1^2} + 3 + \frac{1}{4^2}$,

$A_6 = \frac{1}{1^2} + 6 + \frac{1}{4^2} \cdots$

We have $A_n \leq 3 + \underbrace{\frac{1}{1^2} + \frac{1}{4^2} + \frac{1}{7^2} + \cdots}_{\leq \frac{\pi^2}{6}}$ So $\{A_n\}$ is bounded. So the series is convergent by Dirichlet's test.

Alternating Series Test Of Leibniz: Assume $b_1 \geq b_2 \geq b_3 \geq \cdots$ and $\lim_{n\to\infty} b_n = 0$. Then

$$\sum_{n=1}^{\infty}(-1)^n b_n = -b_1 + b_2 - b_3 + \cdots \quad \text{and} \quad \sum_{n=1}^{\infty}(-1)^{n+1} b_n = b_1 - b_2 + b_3 - \cdots$$

are convergent.

4.66 Example. Let $b_n = \frac{1}{n}$. Then

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln 2$$

is convergent.

The series $\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ i.e. $c_n = \frac{(-1)^{n+1}}{n}$ is convergent but $\sum_{n=1}^{\infty} |c_n| = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

4.67 Definition. Let $\sum c_n$ be a series. If $\sum c_n$ is convergent but $\sum |c_n|$ is divergent, we say $\sum c_n$ is *conditionally convergent*. If $\sum |c_n|$ is also convergent, we say $\sum c_n$ is *absolutely convergent*.

4.68 Theorem. If $\sum |c_n|$ is convergent then $\sum c_n$ is also convergent.

Proof. Use Cauchy criterion. Let $\varepsilon > 0$ be given. Since $\sum |c_n|$ is convergent, there is n_0 such that for all $n \geq m \geq n_0$ we have $|\sum_{k=m}^{n} |c_k|| < \varepsilon$. Let $n \geq m \geq n_0$. Then

$$\left|\sum_{k=m}^{n} c_k\right| \leq \sum_{k=m}^{n} |c_k| < \varepsilon$$

So $\sum c_n$ satisfies Cauchy criterion. Then $\sum c_n$ is convergent. □

For absolute convergence we can use root test, ratio test or comparison test. For conditional convergence we can use Dirichlet's test or alternating series test. (Only for alternating series.)

4.69 Example. Consider

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = S$$

The series is convergent by the alternating series test.

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \left(\left(\frac{1}{4} - \frac{1}{5} \right) + \left(\frac{1}{6} - \frac{1}{7} \right) + \cdots \right) = \frac{5}{6} - (\text{a positive number}) < \frac{5}{6}$$

Consider the rearrangement

$$\underbrace{1 + \frac{1}{3} - \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{7} - \frac{1}{4}} + \underbrace{\frac{1}{9} + \frac{1}{11} - \frac{1}{6}} + \underbrace{\frac{1}{13} + \frac{1}{15} - \frac{1}{8}} + \cdots$$

Each group is in the form

$$\begin{aligned} \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} &= \frac{(4n-1)2n + (4n-3)2n - (4n-3)(4n-1)}{(4n-3)(4n-1)2n} \\ &= \frac{8n^2 - 2n + 8n^2 - 6n - 16n^2 + 4n + 12n - 3}{\cdots} \\ &= \frac{8n-3}{\cdots} > 0 \end{aligned}$$

If t_n is the n-th partial sum of the rearrangement. Then $t_3 < t_6 < t_9 < \cdots$ Then $\limsup_{n\to\infty} t_n > t_3 = 1 + \frac{1}{3} - \frac{1}{2} = \frac{5}{6}$. It follows that $\lim_{n\to\infty} t_n$ cannot be S since $S < \frac{5}{6}$. This is a property of the conditionally convergent series.

Given a conditionally convergent series $\sum a_n$ and $-\infty \le r \le +\infty$, it is possible to find a rearrangement of the series such that rearrangement has sum$= r$.

4.70 Definition. Let $\phi : \mathbb{N} \to \mathbb{N}$ be a 1-1, onto function. Let $\sum_{n=1}^{\infty} a_n$ be a series. Let $b_n = a_{\phi(n)}$. The series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_{\phi(n)}$ is called a *rearrangement* of the series $\sum_{n=1}^{\infty} a_n$.

If

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \\ \sum_{n=1}^{\infty} b_n &= 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots \end{aligned}$$

Then $\phi(1) = 1$, $\phi(2) = 3$, $\phi(3) = 2$, $\phi(4) = 5$, $\phi(5) = 7$, $\phi(6) = 4$, ...

4.71 Theorem. Assume $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, i.e. $\sum_{n=1}^{\infty} |a_n|$ is convergent. Then every rearrangement of $\sum_{n=1}^{\infty} a_n$ is convergent and it converges to the same sum.

Proof. Let $\sum_{n=1}^{\infty} a_{\phi(n)}$ be a rearrangement of $\sum_{n=1}^{\infty} a_n$. Let

$$\begin{aligned} s_n &= a_1 + a_2 + \cdots + a_n \\ t_n &= a_{\phi(1)} + a_{\phi(2)} + \cdots + a_{\phi(n)} \end{aligned}$$

Given $s_n \to s$ where $s \in \mathbb{R}$ ($s \neq \mp\infty$). Show $s_n - t_n \to 0$ as $n \to \infty$. Let $\varepsilon > 0$ be given. Since $\sum a_n$ converges absolutely, there is N such that for all $n \geq m \geq N$ we have $\sum_{k=m}^{n} |a_k| < \varepsilon$. Find $p \geq N$ such that

$$\{1, 2, \ldots, N\} \subset \{\phi(1), \phi(2), \ldots, \phi(p)\}$$

Take $n > p$. Then

$$\begin{aligned} |s_n - t_n - 0| =& |s_n - t_n| \\ =& |a_1 + a_2 + \cdots + a_N + a_{N+1} + \cdots + a_n \\ & \underbrace{\underbrace{-a_{\phi(1)} - a_{\phi(2)} - \cdots - a_{\phi(p)}}_{N \text{ of them will be cancelled}} - \cdots - a_{\phi(n)}|}_{\text{The remaining terms will be of the form } -a_k \text{ where } k>N} \end{aligned}$$

Let $q = \max\{k : \mp a_k \text{ remains in the above}\}$. Then

$$|s_n - t_n - 0| \leq \sum_{k=N+1}^{q} |a_k| < \varepsilon$$

□

4.6 Operations With Series

4.72 Theorem. Let $\sum a_n$, $\sum b_n$ be convergent series with sums A and B. Let $c \in \mathbb{R}$ be a constant. Then

$$\sum (a_n + b_n) \text{ is convergent and has sum } A + B$$
$$\sum c a_n \text{ is convergent and has sum } cA$$

4.73 Remark. The sum of two divergent series may be convergent.

4.74 Example.
$\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} -\frac{1}{n+1}$ are both divergent.
$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent.

4.75 Example. $\sum_{n=0}^{\infty} \left(\left(\frac{2}{3}\right)^n - 5\frac{(-1)^n}{4^n}\right)$ is convergent.

$$\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n - 5\sum_{n=1}^{\infty} \left(-\frac{1}{4}\right)^n = \frac{1}{1-\frac{2}{3}} - 5\frac{1}{1-\left(-\frac{1}{4}\right)} = 3 - 5\frac{4}{5} = -1$$

Cauchy Product Of Two Series: Consider the series

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots \qquad \text{and} \qquad \sum_{n=1}^{\infty} b_n = b_0 + b_1 + b_2 + \cdots$$

We define a new series $\sum_{n=0}^{\infty} c_n$ as follows

$$\begin{aligned} c_0 &= a_0 b_0 \\ c_1 &= a_0 b_1 + a_1 b_0 \\ c_2 &= a_0 b_2 + a_1 b_1 + a_2 b_0 \\ &\vdots \\ c_n &= a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k} \end{aligned}$$

$\sum_{n=0}^{\infty} c_n$ is called the Cauchy product of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$.

4.76 Theorem. Assume

(a) $\sum_{n=0}^{\infty} a_n$ is absolutely convergent and $\sum_{n=0}^{\infty} a_n = A$

(b) $\sum_{n=0}^{\infty} b_n$ is convergent and $\sum_{n=0}^{\infty} b_n = B$

(c) $c_n = \sum_{k=0}^{n} a_k b_{n-k} \quad n = 0, 1, 2, \ldots$

Then $\sum_{n=0}^{\infty} c_n$ is convergent and has sum $C = AB$.

4.77 Remark. The above theorem is not true if both series are conditionally convergent.

4.78 Example. Consider the Cauchy product of the conditionally convergent series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} = \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$$

with itself. We have $a_n = b_n = \frac{(-1)^n}{\sqrt{n+1}}$. Then

$$c_n = \sum_{k=0}^{n} \frac{(-1)^k}{\sqrt{k+1}} \cdot \frac{(-1)^{n-k}}{\sqrt{n-k+1}} = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}}$$

We have

$$(n-k+1)(k+1) = nk+n-k^2-k+k+1 = n+1+k(n-k) \leq n+1+\frac{n^2}{4} = \left(\frac{n}{2}+1\right)^2$$

When $0 \leq x \leq n$, max. of $x(n-x)$ is $\frac{n^2}{4}$ so we have

$$\begin{aligned} |c_n| &= \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}} \geq \sum_{k=0}^{n} \frac{1}{\sqrt{\left(\frac{n}{2}+1\right)}} \\ &= \sum_{k=0}^{n} \frac{1}{\frac{n}{2}+1} = \frac{n+1}{\frac{n}{2}+1} = \frac{2n+2}{n+2} \geq 1 \end{aligned}$$

So "$\lim_{n\to\infty} c_n = 0$" cannot be true. So the series $\sum c_n$ is divergent.

5 Continuity

5.1 General

Let (X, d_X) and (Y, d_Y) be two metric spaces. Let $E \neq \emptyset$ be a non-empty subset of X, $f : E \to Y$, $p \in E'$, $q \in Y$. We say $\lim_{x\to p} f(x) = q$ (or $f(x) \to q$ as $x \to p$) if for every $\varepsilon > 0$ there is $\delta > 0$ with the following property: For every $x \in E$ with $d_X(x, p) < \delta$ we have $d_Y(f(x), q) < \varepsilon$. Equivalently, for every $\varepsilon > 0$ there is $\delta > 0$ such that $f\left(B^X_\delta(p) \cap E\right) \subset B^Y_\varepsilon(q)$. $\delta > 0$ depends in general on $\varepsilon > 0$ and the point p. For $\lim_{x\to p} f(x)$, $f(p)$ need not be defined.

5.1 Example. Let $X = \mathbb{R}^2$ with d_2 and $Y = \mathbb{R}$ with $|\cdot|$ metric.
Let $E = \{(x, y) : (x, y) \in \mathbb{R}^2 \text{ and } xy \neq 0\}$. $f : E \to \mathbb{R}$, $f(x, y) = \frac{x}{y} \sin\left(\frac{y}{x}\right)$
Let $p = (a, 0)$ where $a > 0$. Then $p \in E'$. Show $\lim_{(x,y)\to(a,0)} f(x, y) = 1$. Let $\varepsilon > 0$ be given. Since $\lim_{t\to 0} \frac{\sin t}{t} = 1$, we have $\delta' > 0$ such that for all t with $0 < |t| < \delta'$ we have $\left|\frac{\sin t}{t} - 1\right| < \varepsilon$. Choose $\delta = \frac{a\delta'}{1+\delta'}$. Then $0 < \delta < a$. Show that for all $(x, y) \in E$ with $d_2((x, y), (a, 0)) < \delta$ we have that $|f(x, y)-1| < \varepsilon$. Let $(x, y) \in E$ be such that $d_2((x, y), (a, 0)) < \delta$ i.e.

$$(x - a)^2 + (y - 0)^2 < \delta^2 \Rightarrow |x - a| < \delta \text{ and } |y - 0| < \delta$$

$a - \delta < x < a + \delta$ so $0 < x$. Let $t = \frac{y}{x}$. Then

$$0 < |t| = \frac{|y|}{|x|} = \frac{|y|}{x} < \frac{\delta}{a - \delta} = \frac{\frac{a\delta'}{1+\delta'}}{a - \frac{a\delta'}{1+\delta'}} = \frac{\frac{a\delta'}{1+\delta'}}{\frac{a+a\delta'-a\delta'}{1+\delta'}} = \frac{a\delta'}{a} = \delta'$$

So $t = \frac{y}{x}$ satisfies $0 < |t| < \delta'$, thus

$$\left|\frac{\sin t}{t} - 1\right| < \varepsilon \quad \text{i.e.} \quad \left|\frac{\sin\left(\frac{y}{x}\right)}{\frac{y}{x}} - 1\right| < \varepsilon$$

5.2 Theorem. $\lim_{x\to p} f(x) = q \Leftrightarrow$ for every sequence $\{p_n\}$ in E with $\lim_{n\to\infty} p_n = p$ and $p_n \neq p$ we have that $\lim_{n\to\infty} f(p_n) = q$.

Proof.

($\Rightarrow$): Suppose $\lim_{x\to p} f(x) = q$. Let $\{p_n\}$ be an arbitrary sequence in E such that $\lim_{n\to\infty} p_n = p$ and $p_n \neq p$. Show $\lim_{n\to\infty} f(p_n) = q$. Let $\varepsilon > 0$ be given. Since $\lim_{x\to p} f(x) = q$, there is $\delta > 0$ such that for all $x \in E$ with $d_X(x, p) < \delta$ we have $d_Y(f(x), q) < \varepsilon$. Since $\lim_{n\to\infty} p_n = p$, there is n_0 such that for all $n \geq n_0$, $d_X(p_n, p) < \delta$. Let $n \geq n_0$. Then $d_Y(f(p_n), q) < \varepsilon$ since $x = p_n$ (for $n \geq n_0$) satisfies $x \in E$ and $d_X(x, p) < \delta$.

($\Leftarrow$): Proof by contraposition. Suppose $\lim_{x\to p} f(x) \neq q$. So there is an $\varepsilon_0 > 0$ such that for every $\delta > 0$ there is $x \in E$ such that $d_X(x,p) < \delta$ and $d_Y(f(x),q) \geq \varepsilon_0$.

$$\text{Let } \delta = 1, \text{ find } x = p_1 \in E \text{ s.t. } d_X(p_1,p) < 1 \text{ and } d_Y(f(p_1),q) \geq \varepsilon_0$$
$$\text{Let } \delta = \frac{1}{2}, \text{ find } x = p_2 \in E \text{ s.t } d_X(p_2,p) < \frac{1}{2} \text{ and } d_Y(f(p_2),q) \geq \varepsilon_0$$
$$\vdots$$
$$\text{Let } \delta = \frac{1}{n}, \text{ find } x = p_n \in E \text{ s.t. } d_X(p_n,p) < \frac{1}{n} \text{ and } d_Y(f(p_n),q) \geq \varepsilon_0$$

Then $\{p_n\}$ is a sequence in E such that $\lim_{n\to\infty} p_n = p$ and $\lim_{n\to\infty} f(p_n) \neq q$. □

5.3 Corollary. Let $E \subset X$, $p \in E'$, $f,g : E \to \mathbb{R}$ such that $\lim_{x\to p} f(x) = A$ and $\lim_{x\to p} g(x) = B$. Then

$$\lim_{x\to p}(f(x)+g(x)) = A+B$$
$$\lim_{x\to p} f(x)g(x) = AB$$
$$\lim_{x\to p} \frac{f(x)}{g(x)} = \frac{A}{B} \quad \text{if } B \neq 0$$

5.4 Definition. Let (X,d_X) and (Y,d_Y) be metric spaces. Let $\emptyset \neq E \subset X$, $f : E \to Y$, $p \in E$. We say f is *continuous at the point* p if $\lim_{x\to p} f(x) = f(p)$, i.e. for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $x \in E$ with $d_X(x,p) < \delta$ we have $d_Y(f(x),f(p)) < \varepsilon$. In general δ depends on ε and p. If f is continuous at every point p of E, we say f is *continuous on* E.

5.5 Theorem. Let (X,d_X) and (Y,d_Y) be metric spaces and $f : X \to Y$ (i.e. $E = X$). f is continuous on $X \Leftrightarrow$ for every open set $V \subset Y$, the inverse image $f^{-1}(V)$ is an open set.

Proof.

($\Rightarrow$): Let f be continuous on X. Let $V \subset Y$ be an arbitrary open set. Show $f^{-1}(V)$ is an open set in X. Let $p \in f^{-1}(V)$ be an arbitrary point. Then $f(p) \in V$. V is open, so there is $s > 0$ such that $B^Y_s(f(p)) \subset V$. f is continuous at p. Then for $\varepsilon = s > 0$, we find $\delta > 0$ such that for all x with $d_X(x,p) < \delta$ we have $d_Y(f(x),f(p)) < \varepsilon$. Show $B^X_\delta(p) \subset f^{-1}(V)$. Let $x \in B^X_\delta(p)$, i.e. $d_X(x,p) < \delta \Rightarrow d_Y(f(x),f(p)) < \varepsilon = s \Rightarrow f(x) \in B^Y_s(f(p)) \subset V$. $f(x) \in V$, so $x \in f^{-1}(V)$.

($\Leftarrow$): Suppose $f^{-1}(V)$ is open for every open set $V \subset Y$. Show f is continuous on X, i.e. show f is continuous at every point of X. Let $p \in X$ be an arbitrary point. Let $\varepsilon > 0$ be given. The set $V = B_\varepsilon^Y(f(p))$ is an open set in Y. Then $f^{-1}(V)$ is an open set in X. Also $p \in f^{-1}(V)$. Then there is $\delta > 0$ such that $B_\delta^X(p) \subset f^{-1}(V)$. Let x be such that $d_X(x,p) < \delta$, i.e. $x \in B_\delta^X(p)$. Then $x \in f^{-1}(V)$, i.e. $f(x) \in V$, i.e. $d_Y(f(x), f(p)) < \varepsilon$. □

5.6 Corollary. Let $f : X \to Y$. f is cotinuous on $X \Leftrightarrow$ for every closed set $F \subset Y$ we have that the inverse image $f^{-1}(F)$ is closed in X.

Proof. $F \subset Y$ is closed $\Leftrightarrow$ F^C is open. Using $f^{-1}(F^C) = \left(f^{-1}(F)\right)^C$ and "f is continuous $\Leftrightarrow$ the inverse image of every open set is open" we get the result. □

5.7 Theorem. Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces. $\emptyset \neq E \subset X$, $f : E \to Y$, $g : f(E) \to Z$, $p \in E$. If f is continuous at p and g is continuous at $q = f(p)$ then $g \circ f$ is continuous at p.

Proof. Let $\varepsilon > 0$ be given. Since g is continuous at q, we have a $\delta' > 0$ such that for all $y \in f(E)$ with $d_Y(y,q) < \delta'$ we have $d_Z(g(y), g(q)) < \varepsilon$. Since f is continuous at p, we have a $\delta > 0$ such that for all $x \in E$ with $d_X(x,p) < \delta$ we have $d_Y(f(x), f(p)) < \delta'$. Let $x \in E$ and $d_X(x,p) < \delta$. Then $d_Y(f(x), f(p)) < \delta'$. So $d_Z(g(y), g(q)) < \varepsilon$ i.e. $d_Z(g(f(x)), g(f(p))) < \varepsilon$. □

Let (X, d) be a metric space and $f : X \to \mathbb{R}^k$. $f(x) \in \mathbb{R}^k$, so we have $f(x) = (f_1(x), f_2(x), \ldots, f_k(x))$ where $f_1, f_2, \ldots, f_k : X \to \mathbb{R}$.

5.8 Example. $f : \mathbb{R}^3 \to \mathbb{R}^2$ and $f(x,y,z) = (\underbrace{x^2y+1}_{f_1(x,y,z)}, \underbrace{z^3x-3}_{f_2(x,y,z)}) \in \mathbb{R}^2$.

5.9 Theorem. $f : X \to \mathbb{R}^k$ is continuous on $X \Leftrightarrow f_1, f_2, \ldots, f_k$ are all continuous on X.

In the above example, $f_1, f_2 : \mathbb{R}^3 \to \mathbb{R}$ are continuous (since they are polynomials) we have that $f : \mathbb{R}^3 \to \mathbb{R}^2$ is also continuous.

5.10 Theorem. Let $f, g : X \to \mathbb{R}$ be continuous at the point p. Then $f + g$ and $f \cdot g$ are continuous at p. $\frac{f}{g}$ is continuous at p if $g(p) \neq 0$.

5.11 Example. $X = \mathbb{R}^k$. Fix a coordinate, say j-th coordinate. Define $f : \mathbb{R}^k \to \mathbb{R}$. f is continuous on $\mathbb{R}^k$. $x = (x_1, x_2, \ldots, x_k) \to x_j$. Fix $p = (p_1, p_2, \ldots, p_k)$ in $\mathbb{R}^k$. Show f is continuous at p. Given $\varepsilon > 0$, choose $\delta = \varepsilon$. Let $x \in \mathbb{R}^k$ be any point such that $d_2(x, p) < \delta$. Then

$$\begin{aligned} |f(x) - f(p)| &= |x_j - p_j| = \sqrt{(x_j - p_j)^2} \\ &\leq \sqrt{(x_1 - p_1)^2 + \cdots + (x_k - p_k)^2} = d_2(x, p) < \delta = \varepsilon \end{aligned}$$

If $n_1, n_2, \ldots, n_k$ are non-negative integers then define $g : \mathbb{R}^k \to \mathbb{R}$ by $g(x) = x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$. Then by the theorem, g is continuous on $\mathbb{R}^k$. So every polynomial $P(x) = \sum c_{n_1 \cdots n_k} x_1^{n_1} \cdots x_k^{n_k}$ is continuous on X.

5.12 Example. $P : \mathbb{R}^2 \to \mathbb{R}$ and $P(x, y) = 5x^2 - 7x^3y^4 + 8y^6 + 5xy^2 - 3$ is continuous on $\mathbb{R}^2$.

5.2 Continuity And Compactness

5.13 Theorem. Let $f : X \to Y$ be continuous on X. Let E be a compact subset of X. Then the image $f(E)$ is a compact subset of Y. (Continuous image of a compact set is compact.)

Proof. Let $C = \{G_\alpha : \alpha \in A\}$ be an open cover of $f(E)$, i.e. every G_α is an open set and $f(E) \subset \bigcup_{\alpha \in A} G_\alpha$. Let $V_\alpha = f^{-1}(G_\alpha)$ and $\alpha \in A$. V_α is open for every $\alpha \in A$. Do we have $E \subset \bigcup_{\alpha \in A} V_\alpha$? Let $x \in E$. Then $f(x) \in f(E)$. Then $f(x) \in G_{\alpha_0}$ for some $\alpha_0 \in A$. So $x \in f^{-1}(G_{\alpha_0}) = V_{\alpha_0}$. So $E \subset \bigcup_{\alpha \in A} V_\alpha$. So the collection $C' = \{f^{-1}(G_\alpha) : \alpha \in A\}$ is an open cover of E. Since E is compact, there are $\alpha_1, \ldots, \alpha_n \in A$ such that

$$\begin{aligned} E &\subset f^{-1}(G_{\alpha_1}) \cup \cdots \cup f^{-1}(G_{\alpha_n}) \\ f(E) &\subset f\left(f^{-1}(G_{\alpha_1}) \cup \cdots \cup f^{-1}(G_{\alpha_n})\right) \\ &= f\left(f^{-1}(G_{\alpha_1})\right) \cup \cdots \cup f\left(f^{-1}(G_{\alpha_n})\right) \\ &\subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n} \end{aligned}$$

So C has a finite subcover $\{G_{\alpha_1}, \ldots, G_{\alpha_n}\}$ of $f(E)$. $\square$

In the proof of the following corollary, we will need the following proposition.

5.14 Proposition. Let $S \neq \emptyset$ be a bounded subset of $\mathbb{R}$. Then $\sup S, \inf S \in \overline{S}$.

Proof. For $\sup S$ only. Let $\alpha = \sup S$. Show every neighborhood B of α contains a point s from S. $B = (\alpha - \varepsilon, \alpha + \varepsilon)$. Since $\alpha - \varepsilon < \sup S$, $\alpha - \varepsilon$ cannot be an upper bound for S. So there is an element $s \in S$ such that $\alpha - \varepsilon < S$. Also if $s \in S$ then $s \leq \alpha < \alpha + \varepsilon$. So $\alpha - \varepsilon < s < \alpha + \varepsilon$, i.e. $s \in B$. $\square$

5.15 Corollary. Let (X, d) be a compact metric space and $f : X \to \mathbb{R}$ be continuous on X. Then there are points $p, q \in X$ such that for all $x \in X$ we have $f(p) \leq f(x) \leq f(q)$. (A continuous real valued function on a compact set attains its min.$= f(p)$ and max.$= f(q)$)

Proof. The set $S = f(X) \neq \emptyset$ is a compact subset of $\mathbb{R}$. S is bounded. Then $\sup S \in \overline{S}$. S is closed, i.e. $\overline{S} = S$ so $\sup S = S = f(X)$. That is, there is $q \in X$ such that $\sup S = f(q)$. For all $x \in X$ we have $f(x) \leq \sup S = f(q)$. Similarly, $\inf S \in S$ so $\inf S = f(p)$ for some $p \in X$. $\square$

5.16 Corollary. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Then there are two points $p, q \in [a, b]$ such that for all $x \in [a, b]$ we have $f(p) \leq f(x) \leq f(q)$.

5.17 Theorem. Let X be a compact metric space, Y be an arbitrary metric space, $f : X \to Y$ be continuous, 1-1, onto. Then the inverse function $g = f^{-1} : Y \to X$ is also continuous.

Proof. Show that for every closed set $F \subset X$, the inverse image $g^{-1}(F)$ is a closed set in Y. We have $g^{-1}(F) = f(F)$. X is compact, F is closed so F is compact. f is continuous, so $f(F)$ is compact. So $f(F)$ is closed. $\square$

5.18 Remark. If compactness of X is removed then the theorem is not true.

5.19 Example. $X = [0, 2\pi]$ in $\mathbb{R}$ with $d(x_1, x_2) = |x_1 - x_2|$. $Y = \{(x, y) : (x, y) \in \mathbb{R}^2,\ x^2 + y^2 = 1\}$ with d_2 metric restricted to Y. Define $f : X \to Y$ as $f(t) = (\cos t, \sin t)$. f is continuous, 1-1 and onto. But f^{-1} is not continuous at the point $p = (1, 0)$.

5.3 Continuity And Connectedness

5.20 Theorem. Let X, Y be metric spaces and $f : X \to Y$ be continuous. Assume X is connected. Then $f(X)$ is also connected. (Continuous image of a connected set is connected.)

Proof. Assume $f(X)$ is disconnected. Then there are sets $E, F \subset Y$ such that $f(X) = E \cup F$ and $\overline{E} \cap F = \emptyset$, $E \cap \overline{F} = \emptyset$, $E \neq \emptyset$, $F \neq \emptyset$. Let $A = f^{-1}(E)$ and $B = f^{-1}(F)$. $A \neq \emptyset$. Let $q \in E \subset f(X)$, so $q = f(x)$ for some $x \in X$. Since $f(x) = q \in E$, $x \in f^{-1}(E) = A$. Similarly, $B \neq \emptyset$.

$$X \subset f^{-1}(f(X)) = f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F) = A \cup B$$

Also $A \cup B \subset X$. So $X = A \cup B$. Show $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$. Assume $\overline{A} \cap B \neq \emptyset$. Let $p \in \overline{A} \cap B$. Then $p \in \overline{A}$ and $\underbrace{p \in B = f^{-1}(F)}_{f(p) \in F}$. $p \in \overline{A}$, then there is a sequence $\{p_n\}$ in A such that $p_n \to p$. f is continuous, so $\lim_{n\to\infty} f(p_n) = f(p)$. $p_n \in A = f^{-1}(E) \Rightarrow f(p_n) \in E$. So $f(p)$ is the limit of a sequence in E. It means that $f(p) \in \overline{E}$. So $f(p) \in \underbrace{\overline{E} \cap F}_{\emptyset}$. Contradiction.

So $\overline{A} \cap B = \emptyset$. Then X is the union of the separated non-empty sets A, B. It means that X is disconnected. □

5.21 Corollary (Intermediate Value Theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Assume $f(a)$ and $f(b)$ have different signs. Then there is a point p such that $a < p < b$ and $f(p) = 0$.

Proof. $[a, b]$ is connected $\Rightarrow$ $f([a, b])$ is connected. So $f([a, b]) = [c, d]$ is an interval. The interval $[c, d]$ contains both negative and positive numbers (namely $f(a)$, $f(b)$). So $[c, d]$ contains $y = 0$. So $0 \in f([a, b])$, i.e. there is $p \in [a, b]$ such that $f(p) = 0$. □

5.4 Uniform Continuity

Let (X, d_X) and (Y, d_Y) be two metric spaces. $E \subset X$ and $f : E \to Y$. We say

(i) f is *continuous* on E if for every $p \in E$, for every $\varepsilon > 0$ there is $\delta = \delta(p, \varepsilon) > 0$ such that for all $q \in E$ with $d_X(q, p) < \delta$ we have $d_Y(f(q), f(p)) < \varepsilon$. (In general $\delta > 0$ depends on $\varepsilon > 0$ and the point $p \in E$.)

(ii) f is *uniformly continuous* on E if for every $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that for all points $p, q \in E$ with $d_X(p,q) < \delta$ we have $d_Y(f(p), f(q)) < \varepsilon$. ($\delta$ depends only on ε. The same δ works for all $p \in E$.)

$$\text{Uniform Continuity} \underset{\not\Leftarrow}{\Rightarrow} \text{Continuity}$$

5.22 Example. Let $X = \mathbb{R}$, $Y = \mathbb{R}$, $E = (0,1)$, $d_X = d_Y = |\cdot|$ and let $f : E \to \mathbb{R}$, $f(x) = \frac{1}{x}$

Claim 1: f is continuous on E.

Claim 2: f is not uniformly continuous on E.

1) Let $p \in E$ and $\varepsilon > 0$ be given. Then $0 < p < 1$. Let $\delta = \frac{\varepsilon p^2}{1+\varepsilon p} > 0$. If $q \in E$ such that $|q-p| < \delta$ then

$$|f(p) - f(q)| = \left|\frac{1}{p} - \frac{1}{q}\right| = \frac{|q-p|}{pq} < \frac{\delta}{pq}$$

We have $|q - p| < \delta$, so $p - \delta < q < p + \delta$. We have $\delta < p$, i.e.

$$\frac{\varepsilon p^2}{1+\varepsilon p} < p \Leftrightarrow \varepsilon p^2 < p + \varepsilon p^2$$

So $0 < p - \delta$ and

$$|f(p) - f(q)| < \frac{\delta}{pq} < \frac{\delta}{p(p-\delta)} = \frac{\frac{\varepsilon p^2}{1+\varepsilon p}}{p\left(p - \frac{\varepsilon p^2}{1+\varepsilon p}\right)} = \frac{\frac{\varepsilon p^2}{1+\varepsilon p}}{p\frac{p+\varepsilon p^2 - \varepsilon p^2}{1+\varepsilon p}} = \frac{\varepsilon p^2}{p^2} = \varepsilon$$

So f is continuous at $p \in E$. Since $p \in E$ is arbitrary, f is continuous on E. Note that $\delta = \frac{\varepsilon p^2}{1+\varepsilon p}$ depends on both ε and p. So we are inclined to say that f is not uniformly continuous on E. But maybe by some other calculation, we can find δ depending only on ε.

2) Show that f is not uniformly continuous on E, i.e. δ cannot be found depending only on ε. Assume for $\varepsilon = 1$, we have a $\delta > 0$ such that for all $p, q \in E$ with $|p - q| < \delta$ we have $|f(p) - f(q)| < 1$.

Case 1: $0 < \delta \leq \frac{1}{3}$. Let $p = \delta$ and $q = \delta + \frac{\delta}{2}$. Then $p, q \in E$ and $|p - q| = \frac{\delta}{2} < \delta$. So $|f(p) - f(q)| < 1$.

$$|f(p) - f(q)| = \left|\frac{1}{p} - \frac{1}{q}\right| = \frac{|p-q|}{pq} = \frac{\delta/2}{\delta\frac{3\delta}{2}} = \frac{1}{3\delta} \geq 1$$

Contradiction.

Case 2: $\frac{1}{3} < \delta$. Let $\delta' = \frac{1}{3} < \delta$. As in case 1, let $p = \delta'$ and $q = \delta' + \frac{\delta'}{2}$. Then $|p - q| < \delta' < \delta$ and $|f(p) - f(q)| \geq 1$. Contradiction.

5.23 Example. Let $X = Y = \mathbb{R}$, $E = [2, 5]$, $f : E \to \mathbb{R}$, $f(x) = x^2$. Then f is uniformly continuous on E. Let $\varepsilon > 0$ be given. Let $\delta = \frac{\varepsilon}{10} > 0$. Let $p, q \in E$ be such that $|p - q| < \delta$. Then

$$\begin{aligned} |f(p) - f(q)| &= |p^2 - q^2| = |(p - q)(p + q)| \\ &= |p - q||p + q| = \underbrace{|p - q|}_{<\delta} \underbrace{(p + q)}_{\leq 10} < 10\delta = \varepsilon \end{aligned}$$

5.24 Theorem. Let $f : X \to Y$ be continuous on X and let $E \subset X$ be compact. Then f is uniformly continuous on E.

Proof. Let $\varepsilon > 0$ be given. Given $p \in E$, since f is continuous at p, we have a $\delta = \delta(p, \varepsilon) > 0$ such that for all $q \in E$ with $d_X(q, p) < \delta(p, \varepsilon)$ we have $d_Y(f(q) - f(p)) < \frac{\varepsilon}{3}$

$$\mathscr{C} = \left\{ B_{\frac{\delta(p,\varepsilon)}{3}}(p) : p \in E \right\}$$

Do this for every $p \in E$. Then $\mathscr{C}$ is an open cover of E. Since E is compact, this open cover has a finite subcover

$$\mathscr{C}' = \left\{ B_{\frac{\delta(p_1,\varepsilon)}{3}}(p_1), \ldots, B_{\frac{\delta(p_n,\varepsilon)}{3}}(p_n) \right\}$$

for some finite set $p_1, \ldots, p_n \in E$. So

$$E \subset B_{\frac{\delta(p_1,\varepsilon)}{3}}(p_1) \cup \cdots \cup B_{\frac{\delta(p_n,\varepsilon)}{3}}(p_n)$$

Let $\delta = \min \left\{ \frac{\delta(p_1,\varepsilon)}{3}, \cdots, \frac{\delta(p_n,\varepsilon)}{3} \right\}$. Then $\delta > 0$. Show this $\delta > 0$ has the property in the definition of uniform continuity. Let $p, q \in E$ be two arbitrary points such that $d_X(p, q) < \delta$. We have that $p \in B_{\frac{\delta(p_i,\varepsilon)}{3}}(p_i)$ and $q \in B_{\frac{\delta(p_j,\varepsilon)}{3}}(p_j)$ for some p_i, p_j from $p_1, \ldots, p_n$.

$$\begin{array}{cc} p \in B_{\frac{\delta(p_i,\varepsilon)}{3}}(p_i) & q \in B_{\frac{\delta(p_j,\varepsilon)}{3}}(p_j) \\ \Downarrow & \Downarrow \\ d_X(p, p_i) < \frac{\delta(p_i,\varepsilon)}{3} < \delta(p_i, \varepsilon) & d_X(q, p_j) < \frac{\delta(p_j,\varepsilon)}{3} < \delta(p_j, \varepsilon) \\ \Downarrow & \Downarrow \\ d_Y(f(p), f(p_i)) < \frac{\varepsilon}{3} & d_Y(f(q), f(p_j)) < \frac{\varepsilon}{3} \end{array}$$

Assume $\delta(p_i,\varepsilon) \leq \delta(p_j,\varepsilon)$. Also

$$d_X(p_i,p_j) \leq \underbrace{d_X(p_i,p)}_{<\frac{\delta(p_i,\varepsilon)}{3}} + \underbrace{d_X(p,q)}_{<\delta} + \underbrace{d_X(q,p_j)}_{<\frac{\delta(p_j,\varepsilon)}{3}} < \delta(p_j,\varepsilon) \Rightarrow d_Y(f(p_i),f(p_j)) < \frac{\varepsilon}{3}$$

So we have

$$d_Y(f(p),f(q)) \leq \underbrace{d_Y(f(p),f(p_i))}_{<\frac{\varepsilon}{3}} + \underbrace{d_Y(f(p_i),f(p_j))}_{<\frac{\varepsilon}{3}} + \underbrace{d_Y(f(p_j),f(q))}_{<\frac{\varepsilon}{3}} < \varepsilon \quad \square$$

6 Sequences And Series Of Functions

6.1 General

Consider the following sequence of functions defined for $0 \le x \le 1$.

$$f_1(x) = x,\ f_2(x) = x^2,\ f_3(x) = x^3,\ \ldots\ , f_n(x) = x^n,\ \ldots$$

Fix any x, $0 \le x \le 1$ and consider $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n$ (x: fixed). If $0 \le x \le 1$ then $\lim_{n\to\infty} x^n = 0$. If $x = 1$ then $\lim_{n\to\infty} x^n = 1$. Define

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Then for every fixed x, $0 \le x \le 1$, we have $\lim_{n\to\infty} f_n(x) = f(x)$.

6.1 Definition. Let E be any non-empty set and $f_n : E \to \mathbb{R}$, $n = 1, 2, \ldots$ $f : E \to \mathbb{R}$. We say $f_n \to f$ *pointwise on* E if for every fixed $x \in E$, $\lim_{n\to\infty} f_n(x) = f(x)$, i.e. for every $x \in E$ and for every $\varepsilon > 0$, there is a natural number $N = N(x, \varepsilon)$ such that for all $n \ge N$, $|f_n(x) - f(x)| < \varepsilon$. f is called the pointwise limit of $\{f_n\}$. In the above example, observe that every f_n is continuous but their pointwise limit f is not continuous on the set $E = [0, 1]$. Also every f_n is differentiable on the interval $E = [0, 1]$ but their pointwise limit f is not differentiable on $E = [0, 1]$.

6.2 Example. Consider $f_n(x) = \frac{\sin nx}{\sqrt{n}}$ on $E = \mathbb{R}$, $f(x) = 0$. Then for every fixed $x \in \mathbb{R}$, $\lim_{n\to\infty} f_n(x) = 0 = f(x)$. We have $f_n'(x) = \sqrt{n}\cos nx$ and $f'(x) = 0$. But $\lim_{n\to\infty} f_n'(x) \ne f'(x)$. Take $x = 0$, then $f_n'(0) = \sqrt{n} \nrightarrow f'(0)$.

6.3 Example. On $E = [0, 1]$, consider the following sequence

$$f_n(x) = \begin{cases} 4n^2 x & \text{if } 0 \le x \le \frac{1}{2n} \\ 4n - 4n^2 x & \text{if } \frac{1}{2n} \le x \le \frac{1}{n} \\ 0 & \text{if } \frac{1}{n} \le x \le 1 \end{cases}$$

$\lim_{n\to\infty} f_n(x) = 0$ for every fixed x so $f(x) = 0$. We have

$$\int_0^1 f(x)dx = 0 \quad \text{and} \quad \int_0^1 f_n(x)dx = \frac{1}{2}\cdot\frac{1}{n}\cdot 2n = 1 \quad \text{so} \quad \lim_{n\to\infty}\int_0^1 f_n(x)dx = 1$$

So we have

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx \ne \int_0^1 \lim_{n\to\infty} f_n(x)dx$$

6.4 Example. $f_n(x) = \frac{x^2}{(1+x^2)^n}$ and $E = \mathbb{R}$. Consider $f(x) = \sum_{n=0}^{\infty} f_n(x)$ where $n = 0, 1, 2, \ldots$

$$\begin{aligned} f(x) &= x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \cdots + \frac{x^2}{(1+x^2)^n} + \cdots \\ &= x^2 \underbrace{\left(1 + \frac{1}{1+x^2} + \left(\frac{1}{1+x^2}\right)^2 + \cdots + \left(\frac{1}{1+x^2}\right)^n + \cdots\right)}_{\text{geometric series with } r=\frac{1}{1+x^2}} \\ &= x^2 \frac{1}{1 - \frac{1}{1+x^2}} = 1 + x^2 \quad \text{if } x \neq 0 \end{aligned}$$

If $x = 0$ then $f(0) = 0 + 0 + \cdots = 0$. So

$$f(x) = \begin{cases} 1 + x^2 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

So the sum $f(x)$ of continuous functions $\sum f_n(x)$ is not continuous on $\mathbb{R}$. Pointwise convergence is not strong enough for the calculus of limits of sequences of functions.

6.2 Uniform Convergence

6.5 Definition. Let E be any non-empty set and $f_n : E \to \mathbb{R}$, $n = 1, 2, \ldots$, $f : E \to \mathbb{R}$ be functions. We say $f_n \to f$ *uniformly on* E if for every $\varepsilon > 0$ we have $N = N(\varepsilon)$ such that for all $n \geq N(\varepsilon)$ and for all $x \in E$ we have $|f_n(x) - f(x)| < \varepsilon$. Here $N = N(\varepsilon)$ depends on ε only and it works for every $x \in E$.

6.6 Example. Let $0 < c < 1$ be a fixed constant. Let $E = [0, c]$, $f_n(x) = x^n$. We have for every fixed $x \in E$, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0$. So $f(x) = 0$, i.e. $x^n \to 0$ pointwise on E. Does $x_n \to 0$ uniformly on E ? Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} c^n = 0$, we have N such that $c^N < \varepsilon$. Let $n \geq N$, $x \in E$

$$|f_n(x) - f(x)| = |x^n - 0| = x^n \leq c^n \leq c^N < \varepsilon$$

6.7 Example. $E = [0, 1)$, $f_n(x) = x^n$. For every fixed x with $0 \leq x < 1$, we have $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0$. So $f(x) = 0$ and $f_n \to f$, i.e. $x^n \to 0$ pointwise on $E = [0, 1)$. But this convergence is not uniform. Assume $f_n \to f$

uniformly on E. Then for $\varepsilon = \frac{1}{4}$ we can find N_1 such that for all $n \geq N_1$ and for all $x \in E = [0,1)$ we have $|f_n(x) - f(x)| < \frac{1}{4}$ i.e. $x_n < \frac{1}{4}$. Also

$$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = e \Rightarrow \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{e}$$

So for $\varepsilon = \frac{1}{e} - \frac{1}{3} > 0$ we have N_2 such that for all $n \geq N_2$ we have

$$\left|\left(\frac{n}{n+1}\right)^n - \frac{1}{e}\right| < \frac{1}{e} - \frac{1}{3} \Rightarrow -\frac{1}{e} + \frac{1}{3} < \left(\frac{n}{n+1}\right)^n - \frac{1}{e} < \frac{1}{e} - \frac{1}{3}$$
$$\Rightarrow \frac{1}{3} < \left(\frac{n}{n+1}\right)^n \quad \text{for all } n \geq N_2$$

Let $N = \max\{N_1, N_2\}$, $x = \frac{N}{N+1}$ and $x \in E$. Since $N \geq N_1$, we have $x^N < \frac{1}{4}$ and since $N \geq N_2$, we have $\frac{1}{3} < x^N$. So $\frac{1}{3} < \left(\frac{N}{N+1}\right)^N < \frac{1}{4}$ i.e. $\frac{1}{3} < \frac{1}{4}$ which is not true.

Cauchy Criterion For Uniform Convergence

Let $E \neq \emptyset$, $f_n : E \to \mathbb{R}$, $n = 1, 2, \ldots$ Assume for every $\varepsilon > 0$ there is a natural number $N = N(\varepsilon)$ such that for all $n, m \geq N(\varepsilon)$ and for all $x \in E$ we have $|f_n(x) - f_m(x)| < \varepsilon$. Then there is a function $f : E \to \mathbb{R}$ such that $f_n \to f$ uniformly on E.

If we have a series of functions $\sum_{n=1}^{\infty} f_n(x)$ defined on a set E, we define $s_n(x) = f_1(x) + \cdots + f_n(x)$. If there is a function $f : E \to \mathbb{R}$ such that $s_n \to f$ uniformly on E then we say the series $\sum_{n=1}^{\infty} f_n(x) = f(x)$ uniformly on E.

Cauchy Criterion: Assume for every $\varepsilon > 0$, there is a natural number $N = N(\varepsilon)$ such that for all $n, m \geq N(\varepsilon)$ with $n \geq m$ and for all $x \in E$ we have $|\sum_{k=m}^{n} f_k(x)| < \varepsilon$. Then there is a function $f : E \to \mathbb{R}$ such that $\sum_{n=1}^{\infty} f_n(x) = f(x)$ uniformly on E.

Weierstrass M-Test: Let $f_n : E \to \mathbb{R}$, $n = 1, 2, \ldots$ Assume for every n there is a number $M_n > 0$ such that

(i) $|f_n(x)| \leq M_n$ for all $x \in E$

(ii) $\sum_{n=1}^{\infty} M_n$ is convergent

Then the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to some function $f(x)$ on E.

6.8 Example. $E = \mathbb{R}$. Consider $\sum_{n=1}^{\infty} \frac{\cos(2nx)}{(2n-1)(2n+1)}$. Then $f_n(x) = \frac{\cos(2nx)}{(2n-1)(2n+1)}$

$$|f_n(x)| = \frac{|\cos(2nx)|}{(2n-1)(2n+1)} \leq \frac{1}{(2n-1)(2n+1)} = M_n \quad \text{for all } x \in E$$

$\sum_{n=1}^{\infty} M_n$ is convergent since $0 < M_n \leq \frac{1}{n^2}$. So there is a function $f : \mathbb{R} \to \mathbb{R}$ such that $\sum_{n=1}^{\infty} f_n(x) = f(x)$ uniformly on $\mathbb{R}$.

6.9 Example. Consider $\sum_{n=1}^{\infty} \frac{x}{n+n^2x^2}$ and $E = [0, +\infty)$. We have

$$f_n(x) = |f_n(x)| = \frac{x}{n + n^2x^2}$$

To find M_n we use calculus. Find max. of $f_n(x)$ for $x \geq 0$.

$$f_n'(x) = \frac{n + n^2x^2 - xn^2 2x}{(n + n^2x^2)^2} = \frac{n - n^2x^2}{(n + n^2x^2)^2} = 0 \Rightarrow x^2 = \frac{1}{n} \Rightarrow x = \frac{1}{\sqrt{n}}$$

$0 \leq x \leq \frac{1}{\sqrt{n}} \Rightarrow x^2 \leq \frac{1}{n} \Rightarrow n^2x^2 \leq n \Rightarrow 0 \leq n - n^2x^2 \Rightarrow f_n'(x) \geq 0$

$\frac{1}{\sqrt{n}} \leq x \Rightarrow \frac{1}{n} \leq x^2 \Rightarrow n \leq n^2x^2 \Rightarrow n - n^2x^2 \leq 0 \Rightarrow f_n'(x) \leq 0$

So $f_n(x)$ has its max. at the point $x = \frac{1}{\sqrt{n}}$.

$$M_n = f_n\left(\frac{1}{\sqrt{n}}\right) = \frac{\frac{1}{\sqrt{n}}}{n + n^2\frac{1}{n}} = \frac{1}{2n^{3/2}}$$

$\sum M_n = \frac{1}{2} \sum \frac{1}{n^{3/2}}$ is convergent. So there is a function $f : E \to \mathbb{R}$ such that $\sum_{n=1}^{\infty} \frac{x}{n+n^2x^2} = f(x)$ uniformly on the set $E = [0, +\infty)$.

6.10 Example. Consider a power series $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$ Assume it has radius of convergence $R > 0$. If $x = \limsup_{n\to\infty} \sqrt[n]{|c_n|}$ then $R = \frac{1}{\alpha}$. Let $0 < r < R$ and $E = [-r, r]$. $f_n(x) = c_n x^n$. For all $x \in E$

$$|f_n(x)| = |c_n||x|^n \leq \underbrace{|c_n| r^n}_{M_n}$$

Is $\sum M_n$ convergent ? Use root test.

$$\limsup_{n\to\infty} \sqrt[n]{|M_n|} = \limsup_{n\to\infty} \sqrt[n]{|c_n|} r = r \underbrace{\limsup_{n\to\infty} \sqrt[n]{|c_n|}}_{\alpha} = r\alpha < R\alpha = 1$$

So by the root test, $\sum M_n$ is convergent. So the power series $\sum c_n x^n$ converges uniformly on $E = [-r, r]$ where $0 < r < R$.

6.3 Uniform Convergence And Continuity

6.11 Theorem. Let (X, d) be a metric space and $E \neq \emptyset$ subset of X. $f_n : E \to \mathbb{R}$, $n = 1, 2, \ldots$ and $f : E \to \mathbb{R}$. Assume $f_n \to f$ uniformly on E. Let x_0 be a limit point of E and assume for every n, $\lim_{x\to x_0} f_n(x) = A_n$. Then $\{A_n\}$ is convergent and $\lim_{x\to x_0} f(x) = \lim_{n\to\infty} A_n$. That is

$$\lim_{x\to x_0} \underbrace{\lim_{n\to\infty} f_n(x)}_{f(x)} = \lim_{n\to\infty} \underbrace{\lim_{x\to x_0} f_n(x)}_{A_n}$$

The two limits can be interchanged.

Proof. Show $\{A_n\}$ is a Cauchy sequence in $\mathbb{R}$. Given $\varepsilon > 0$, find $N = N(\varepsilon)$ such that for all $n \geq N(\varepsilon)$ and for all $x \in E$, $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$. Let $n, m \geq N(\varepsilon)$. Then for any $x \in E$

$$|f_n(x) - f_m(x)| \leq \underbrace{|f_n(x) - f(x)|}_{<\frac{\varepsilon}{2}} + \underbrace{|f(x) - f_m(x)|}_{<\frac{\varepsilon}{2}} < \varepsilon$$

This proof shows uniformly convergent $\Rightarrow$ uniformly Cauchy

Take $n, m \geq N(\varepsilon)$ and fix them. For every $x \in E$ we have $|f_n(x) - f_m(x)| < \varepsilon$. Let $x \to x_0$. $|A_n - A_m| \leq \varepsilon$. True for all $n, m \geq N(\varepsilon)$. So $\{A_n\}$ is Cauchy. Since $\mathbb{R}$ is complete, $\lim_{n\to\infty} A_n = A$ exists in $\mathbb{R}$. To show $\lim_{x\to x_0} f(x) = A$, let $\varepsilon > 0$ be given. $f_n \to f$ uniformly on E, so there is $N_1 = N_1(\varepsilon)$ such that for all $n \geq N_1(\varepsilon)$ and for all $x \in E$

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3} \cdots (1)$$

$A_n \to A$, so there is $N_2 = N_2(\varepsilon)$ such that for all $n \geq N_2(\varepsilon)$ we have

$$|A_n - A| < \frac{\varepsilon}{3} \cdots (2)$$

Let $N = \max\{N_1(\varepsilon), N_2(\varepsilon)\}$. Since $\lim_{x\to x_0} f_N(x) = A_N$, we have $\delta > 0$ such that for all $x \in E$ with $d_X(x, x_0) < \delta$ we have

$$|f_N(x) - A_N| < \frac{\varepsilon}{3} \cdots (3)$$

Let $x \in E$ and $d_X(x, x_0) < \delta$. Then

$$|f(x) - A| \leq \underbrace{|f(x) - f_N(x)|}_{<\frac{\varepsilon}{3} \text{ by } (1)} + \underbrace{|f_N(x) - A_N|}_{<\frac{\varepsilon}{3} \text{ by } (3)} + \underbrace{|A_N - A|}_{<\frac{\varepsilon}{3} \text{ by } (2)} < \varepsilon \qquad \square$$

6.12 Corollary. Let (X, d) be a metric space. Let $f_n : X \to \mathbb{R}$, $n = 1, 2, \ldots$ $f : X \to \mathbb{R}$. Assume $f_n \to f$ uniformly on X and each f_n is continuous on X. Then f is also continuous on X. (Uniform limit of continuous functions is continuous.)

Proof. Fix $x_0 \in X$. Show $\lim_{x \to x_0} f(x) = f(x_0)$.

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \lim_{x \to x_0} f_n(x) = \lim_{n \to \infty} f_n(x_0) = f(x_0) \qquad \square$$

6.13 Remark. If each f_n is uniformly continuous on X and $f_n \to f$ uniformly on X then f is also uniformly continuous on X.

6.14 Example. $E = [0, 1]$, $f_n(x) = x^n$, $n = 1, 2, \ldots$

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Each f_n is continuous on E but f is not continuous. So $\{f_n\}$ does not converge to f uniformly.

$$\underbrace{\lim_{x \to 1^-} \underbrace{\lim_{n \to \infty} x^n}_{0}}_{0} \neq \underbrace{\lim_{n \to \infty} \underbrace{\lim_{x \to 1^-} x^n}_{1}}_{1}$$

(1) x fixed. Take limit as $n \to \infty$
(2) Take limit as $x \to 1^-$

(1) n fixed. Take limit as $x \to 1^-$
(2) Take limit as $n \to \infty$

6.15 Corollary. Let (X, d) be a metric space. Assume $f_n : X \to \mathbb{R}$, $n = 1, 2, \ldots$ is continuous on X for every n and $\sum_{n=1}^{\infty} f_n(x) = f(x)$ uniformly on X. Then f is also continuous on X.

$$\begin{array}{ccc} \lim_{x \to x_0} \sum_{n=1}^{\infty} f_n(x) & \text{and} & \sum_{n=1}^{\infty} \lim_{x \to x_0} f_n(x) \\ \| & & \| \\ \lim_{x \to x_0} f(x) & & \sum_{n=1}^{\infty} f_n(x_0) \\ \| & & \| \\ f(x_0) & \overset{\text{equal}}{=} & f(x_0) \end{array}$$

So $\lim_{x \to x_0} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \lim_{x \to x_0} f_n(x)$ if $\sum f_n(x) = f(x)$ uniformly on X.

6.16 Example. Consider

$$\sum_{n=1}^{\infty} x(1-x)^n = x(1-x) + x(1-x)^2 + x(1-x)^3 + \cdots$$
$$= x(1-x)\underbrace{\left[1 + (1-x) + (1-x)^2 + \cdots\right]}_{\text{geometric series with } r=1-x}$$

Also for $x = 0$ we have $0 + 0 + \cdots$ so let $f(x) = \sum_{n=1}^{\infty} x(1-x)^n$. Then

$$f(x) = \begin{cases} 1-x & \text{if } 0 < x < 2 \\ 0 & \text{if } x = 0 \end{cases}$$

So $E = [0, 2)$. Do we have uniform convergence on $E = [0, 2)$?

$$\lim_{x \to 0^+} \sum_{n=1}^{\infty} x(1-x)^n \stackrel{?}{=} \sum_{n=1}^{\infty} \lim_{x \to 0^+} x(1-x)^n$$

LHS$=\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(1-x) = 1$
RHS$=\sum_{n=1}^{\infty} \lim_{x\to 0^+} x(1-x)^n = 0 + 0 + \cdots = 0$
$1 \neq 0$ so convergence is not uniform.

6.17 Example. $\lim_{x\to 0^+} \sum_{n=1}^{\infty} \frac{nx^2}{n^3+x^3} =$? Take $E = [0, 1]$.

$$f_n(x) = |f_n(x)| = \frac{nx^2}{n^3 + x^3} \leq \frac{n}{n^3} = \frac{1}{n^2} = M_n$$

$\sum M_n$ is convergent so by Weierstrass M-test, $\sum_{n=1}^{\infty} \frac{nx^2}{n^3+x^3}$ converges to some $f(x)$ uniformly on E. Then

$$\lim_{x\to 0^+} \sum_{n=1}^{\infty} \frac{nx^2}{n^3 + x^3} = \sum_{n=1}^{\infty} \lim_{x\to 0^+} \frac{nx^2}{n^3 + x^3} = \sum_{n=1}^{\infty} 0 = 0 + 0 + \cdots = 0$$

6.18 Example. Let $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$ be a power series with radius of convergence $R > 0$. Then for all x with $-R < x < R$, the power series converges. Let x_0 be such that $-R < x_0 < R$. Do we have

$$\lim_{x\to x_0} \sum_{n=0}^{\infty} c_n x^n \stackrel{?}{=} \sum_{n=0}^{\infty} \lim_{x\to x_0} c_n x^n$$

Find $r > 0$ such that $-R < -r < x_0 < r < R$. If $E = [-r, r]$ then the power series converges uniformly on E and $x_0 \in E$. So

$$\lim_{x \to x_0} \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \lim_{x \to x_0} c_n x^n$$

So given a power series $\sum_{n=0}^{\infty} c_n x^n$ with $R > 0$ and given any x_0 such that $-R < x_0 < R$, we have

$$\lim_{x \to x_0} \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \lim_{x \to x_0} c_n x^n = \sum_{n=0}^{\infty} c_n x_0^n$$

6.4 Uniform Convergence And Integration

6.19 Theorem. Assume $f_n : [a, b] \to \mathbb{R}$, $n = 1, 2, \ldots$ are integrable on $[a, b]$ (continuous functions are integrable) and $f_n \to f$ uniformly on $[a, b]$ for some $f : [a, b] \to \mathbb{R}$. Then f is also integrable on $[a, b]$ and

$$\int_a^b f(x)dx = \lim_{n \to \infty} \int_a^b f_n(x)dx$$

Proof. We omit the integrability proof. Let $\varepsilon > 0$ be given. Since $f_n \to f$ uniformly on $[a, b]$, we have N such that for all $n \geq N$, for all $a \leq x \leq b$, $|f_n(x) - f(x)| < \varepsilon'$. Let $n \geq N$. Then

$$\begin{aligned} \left| \int_a^b f_n(x)dx - \int_a^b f(x)dx \right| &= \left| \int_a^b (f_n(x) - f(x))dx \right| \\ &\leq \int_a^b |f_n(x) - f(x)|dx \\ &\leq \int_a^b \varepsilon' dx = \varepsilon'(b - a) \\ &< \underbrace{2\varepsilon'(b - a)}_{\varepsilon} \end{aligned}$$

So let $\varepsilon' = \frac{\varepsilon}{2(b-a)}$ □

6.20 Example. $E = [0, 1]$, $f_n(x) = n^2 x^n (1 - x)$. Let $x \in E$ be fixed.

If $x = 0$ then $f_n(0) = 0 \to 0$

If $x = 1$ then $f_n(1) = 0 \to 0$

If $x = 0 < x < 1$ then $f_n(x) = n^2x^n(1-x) \to 0$

So $f(x) = 0$ for all $0 \leq x \leq 1$. Do we have $f_n \to f$ uniformly on $[0,1]$?

$$\int_0^1 f(x)dx \stackrel{?}{=} \lim_{n\to\infty} \int_0^1 f_n(x)dx$$

For LHS we have

$$\int_0^1 f(x)dx = \int_0^1 0dx = 0$$

For RHS we have

$$\begin{aligned}
\lim_{n\to\infty} \int_0^1 f_n(x)dx &= \lim_{n\to\infty} \left(\int_0^1 n^2x^n dx - \int_0^1 n^2x^{n+1}dx \right) \\
&= \lim_{n\to\infty} \left(n^2 \frac{x^{n+1}}{n+1}\Big|_0^1 - n^2 \frac{x^{n+2}}{n+2}\Big|_0^1 \right) \\
&= \lim_{n\to\infty} n^2 \left(\frac{1}{n+1} - \frac{1}{n+2} \right) \\
&= \lim_{n\to\infty} \frac{n^2}{(n+1)(n+2)} = 1
\end{aligned}$$

$0 \neq 1$ so convergence is not uniform.

6.21 Corollary. Assume $f_n : [a,b] \to \mathbb{R}$, $n = 1, 2, \ldots$ are integrable on $[a,b]$ and the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[a,b]$. Then $\sum_{n=1}^{\infty} f_n(x)$ is also integrable on $[a,b]$ and

$$\int_a^b \left(\sum_{n=1}^{\infty} f_n(x) \right) dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)dx$$

6.22 Example. Consider $F(x) = \sum_{n=1}^{\infty} \frac{x}{n(x+n)}$ where $0 \leq x \leq 1$. Show the series converges uniformly on $E = [0,1]$.

$$f_n(x) = |f_n(x)| = \frac{x}{n(x+n)} \leq \frac{1}{n^2} = M_n$$

$\sum M_n$ is convergent. So by Weierstrass M-test, the series $\sum_{n=1}^{\infty} \frac{x}{n(x+n)}$ is uniformly convergent on $E = [0,1]$. Let us call

$$\int_0^1 F(x)dx = \gamma$$

Then we have

$$\begin{aligned}
\gamma &= \sum_{n=1}^{\infty} \int_0^1 \frac{x}{n(x+n)} dx = \sum_{n=1}^{\infty} \int_0^1 \left(\frac{1}{n} - \frac{1}{x+n} \right) dx \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{n} x - \ln(x+n) \right) \Big|_0^1 = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \ln(1+n) + \ln n \right) \\
&= \lim_{n\to\infty} \left(\sum_{k=1}^{n} \left(\frac{1}{k} - \ln(1+k) + \ln k \right) \right) \\
&= \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln 2 + \ln 1 - \ln 3 + \ln 2 - \cdots - \ln(1+n) + \ln n \right) \\
&= \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln(n+1) \right)
\end{aligned}$$

Let us define

$$\alpha_n = \sum_{k=1}^{n} \frac{1}{k} - \ln(n+1)$$

Then $\lim_{n\to\infty} \alpha_n = \gamma$.

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k} &= \alpha_n + \ln(n+1) \\
&= \alpha_n + \ln n + \ln(n+1) - \ln n \\
&= \gamma + \ln n + \underbrace{\ln(n+1) - \ln n + \alpha_n - \gamma}_{\text{call } \sigma_n} \\
&= \gamma + \ln n + \sigma_n
\end{aligned}$$

So $\sum_{k=1}^{n} = \ln n + \gamma + \sigma_n$ where $\sigma_n \to 0$ as $n \to \infty$. γ is called *Euler's constant.* $\gamma \approx 0.57721$. It is not known whether γ is rational or irrational. So for large n, $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \approx \ln n + \gamma$.

6.23 Example. Let $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$ be a power series with radius of convergence $R > 0$. Take any x_0 such that $-R < x_0 < R$. Then

$$\int_0^{x_0} \left(\sum_{n=0}^{\infty} c_n x^n \right) dx = \sum_{n=0}^{\infty} \int_0^{x_0} c_n x^n dx$$

6.24 Example. $\sum_{n=1}^{\infty} \frac{1}{2^n n} = ?$ Consider $\sum_{n=0}^{\infty} x^n$ with $R = 1$. Take $x_0 = \frac{1}{2}$.

$$\int_0^{1/2} \left(\sum_{n=0}^{\infty} x^n \right) dx = \int_0^{1/2} \frac{1}{1-x} dx = -\ln(1-x) \Big|_0^{1/2} = -\ln \frac{1}{2} = \ln 2$$

$$\sum_{n=0}^{\infty} \int_0^{1/2} x^n dx = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \Big|_0^{1/2} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}(n+1)} = \sum_{n=1}^{\infty} \frac{1}{2^n n}$$

We know that

$$\int_0^{1/2} \left(\sum_{n=0}^{\infty} x^n \right) dx = \sum_{n=0}^{\infty} \int_0^{1/2} x^n dx$$

So we get

$$\sum_{n=1}^{\infty} \frac{1}{2^n n} = \ln 2$$

6.5 Uniform Convergence And Differentiation

6.25 Theorem. Let $f_n : [a,b] \to \mathbb{R}$, $n = 1, 2, \dots$ be differentiable functions. Assume

(i) $\{f_n'\}$ converges uniformly to some function g on $[a,b]$.

(ii) There is $x_0 \in [a,b]$ such that $\{f_n(x_0)\}$ is convergent.

Then there is a differentiable function $f : [a,b] \to \mathbb{R}$ such that $f_n \to f$ uniformly on $[a,b]$ and $f'(x) = g(x)$ for all $x \in [a,b]$.

Proof. $\{f_n\}$ is uniformly convergent on $[a,b]$. Use Cauchy criterion. Let $\varepsilon > 0$ be given. Let $\varepsilon' = \cdots$ Find N_1 such that for all $n, m \geq N_1$, $|f_n(x_0) - f_m(x_0)| < \varepsilon'$. Find N_2 such that for all $n, m \geq N_2$ and for all $x \in [a,b]$, $|f_n'(x) - f_m'(x)| < \varepsilon'$. Let $N = \max\{N_1, N_2\}$ and $n, m \geq N$. Take $x \in [a,b]$. Apply Mean Value Theorem to $f_n - f_m$ on the interval $[x_0, x]$ (or $[x, x_0]$). Then there is a point t between x_0 and x

$$f_n(x) - f_m(x) - (f_n(x_0) - f_m(x_0)) = (f_n'(t) - f_m'(t))(x - x_0)$$
$$|f_n(x) - f_m(x) - (f_n(x_0) - f_m(x_0))| = \underbrace{|f_n'(t) - f_m'(t)|}_{<\varepsilon'} \underbrace{|x - x_0|}_{\leq b-a} < \varepsilon'(b-a)$$

Then we have

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq |f_n(x) - f_m(x) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)| \\ &< \varepsilon'(b-a) + \varepsilon' = \varepsilon'(b-a+1) \end{aligned}$$

So let $\varepsilon' = \frac{\varepsilon}{b-a+1}$. Then there is a function $f : [a,b] \to \mathbb{R}$ such that $f_n \to f$ uniformly on $[a,b]$ and $f'(x) = g(x)$ for all $x \in [a,b]$. $\square$

6.26 Corollary. Let $f_n : [a,b] \to \mathbb{R}$, $n = 1, 2, \ldots$ be differentiable functions. Assume

(i) $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly on $[a,b]$.

(ii) There is $x_0 \in [a,b]$ such that $\sum_{n=1}^{\infty} f_n(x_0)$ is convergent.

Then the series $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent on $[a,b]$ and

$$\left(\sum_{n=1}^{\infty} f_n(x)\right)' = \sum_{n=1}^{\infty} f_n'(x)$$

6.27 Example. Let the power series $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$ have radius of convergence $R > 0$. Then $R = \frac{1}{\alpha}$ where $\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}$. Here $f_n(x) = c_n x^n$.

$$\sum_{n=0}^{\infty} f_n'(x) = \sum_{n=1}^{\infty} c_n n x^{n-1} = \sum_{n=0}^{\infty} c_{n+1}(n+1)x^n$$

For this series

$$\begin{aligned}\limsup_{n\to\infty} \sqrt[n]{|c_{n+1}(n+1)|} &= \limsup_{n\to\infty} \left(\sqrt[n+1]{|c_{n+1}(n+1)|}\right)^{\frac{n+1}{n}} \\ &= \limsup_{n\to\infty} \left(\sqrt[n+1]{|c_{n+1}|}\right)^{\frac{n+1}{n}} \left(\sqrt[n+1]{n+1}\right)^{\frac{n+1}{n}} = \alpha\end{aligned}$$

So the differentiated series $\sum_{n=1}^{\infty} c_n n x^{n-1}$ and the original series $\sum_{n=0}^{\infty} c_n x^n$ have the same R. So if $0 < r < R$ then $\sum_{n=1}^{\infty} c_n n x^{n-1}$ converges uniformly on $[-r, r]$. Then for all x with $-r \le x \le r$ we have

$$\left(\sum_{n=0}^{\infty} c_n x^n\right)' = \sum_{n=1}^{\infty} c_n n x^{n-1}$$

Given any x with $-R < x < R$, we can find a number r such that $0 < r < R$ and $-r \le x \le r$. So

$$\left(\sum_{n=0}^{\infty} c_n x^n\right)' = \sum_{n=1}^{\infty} c_n n x^{n-1}$$

is true for all x with $-R < x < R$.

6.28 Remark. This result is not true for a general series of functions.

6.29 Example. $\sum_{n=1}^{\infty} \frac{n}{2^n} = ?$ Let $x = \frac{1}{2}$. Then we have

$$\sum_{n=1}^{\infty} nx^n = x \sum_{n=1}^{\infty} nx^{n-1} = x \left(\sum_{n=1}^{\infty} x^n \right)' = \left(\frac{1}{1-x} \right)' = \frac{1}{(1-x)^2}$$

for $-1 < x < 1$. So we have

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = \frac{\frac{1}{2}}{\frac{1}{4}} = 2$$

■ **THE END** ■

7 Figures

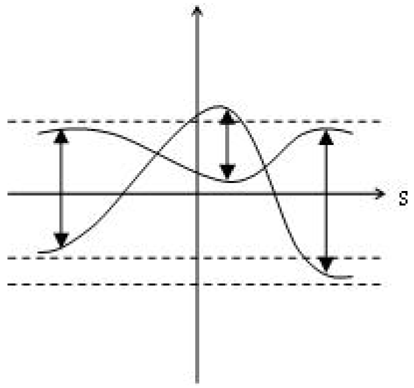


Figure 1: Vertical distances between two graphs

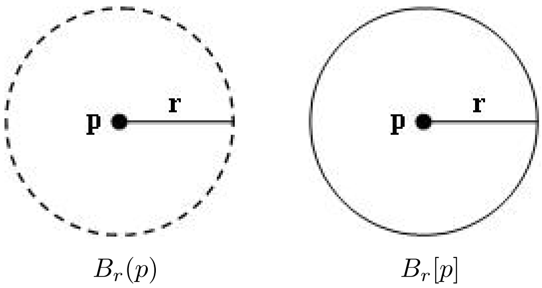


Figure 2: Open and closed balls with d_2 metric

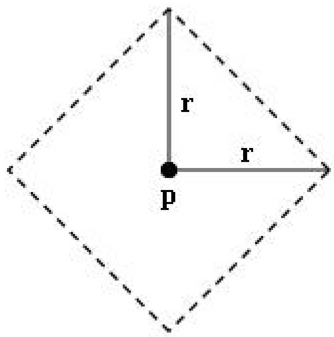


Figure 3: $B_r(p)$ with d_1 metric

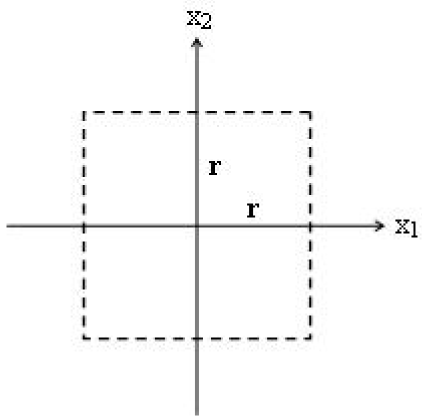


Figure 4: $B_r(\underline{0})$ with d_∞ metric

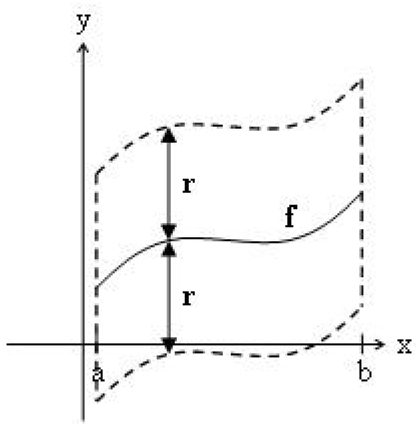


Figure 5: $B_r(f)$

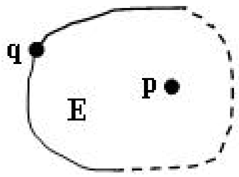
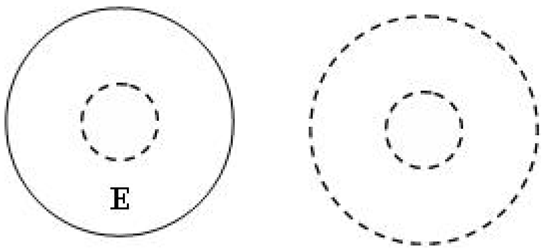


Figure 6: p is an interior point of E but q is not



The set E $\qquad$ $\operatorname{int} E$

Figure 7: The set E and $\operatorname{int} E$

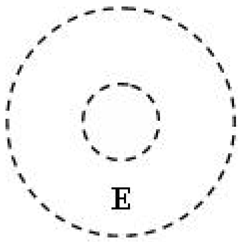


Figure 8: E is an open set

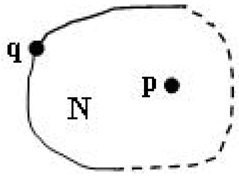
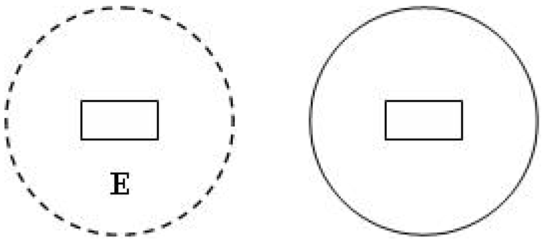
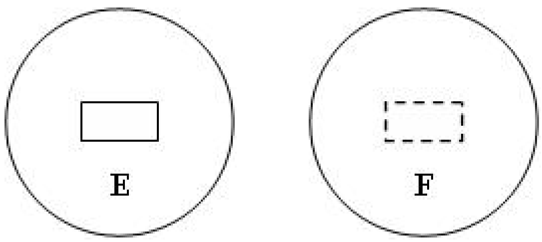


Figure 9: N is a neighborhood of p but not a neighborhood of q



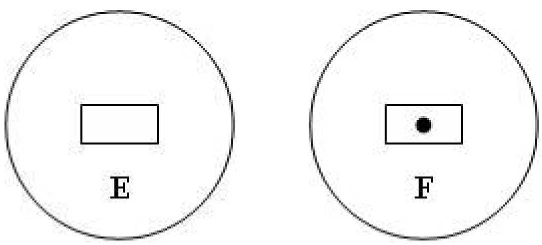
The set E Set of limit points of E

Figure 10: The set E and its limit points



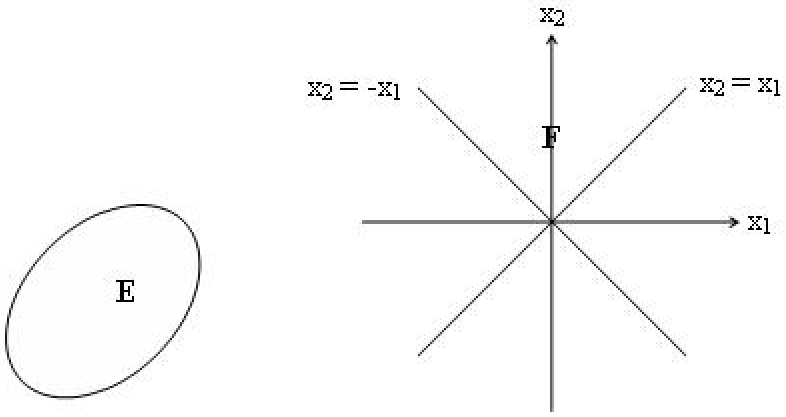
E is closed F is not closed and not open

Figure 11: A set may be neither closed nor open



E is perfect F is not perfect

Figure 12: E is perfect but F has an isolated point



E is bounded $\qquad$ F is unbounded

Figure 13: Bounded and unbounded sets

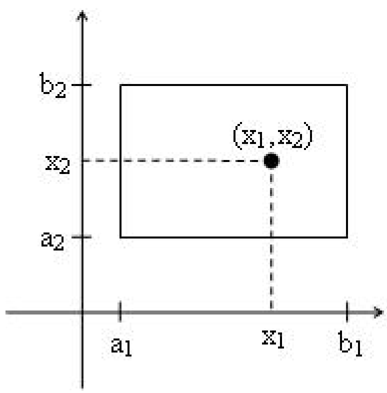


Figure 14: A k-cell in $\mathbb{R}^2$